Light Condensed Anima and Étale Sheaves

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These are the notes for a talk I gave in the *Kleine Topologie Seminar* at the University of Münster in the Winter of 2025. The goal of the talk was to introduce light condensed anima, illustrate their theory of étale sheaves and discuss some examples. The material is taken from [Cla25, Sections 4-5], and supplementary notes on sites and sheaves can be found at [Shv].

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1 Light Condensed Anima

In this section, we define light condensed anima.

Definition 1.1. A *light profinite set* is a topological space S that can be written as a cosequential limit of finite discrete spaces. The full subcategory of Top on the light profinite spaces is denoted $\mathrm{ProfSet}^{\mathrm{light}}$.

Remark 1.2. This subcategory is closed under countable limits, closed subspaces and finite coproducts.

Remark 1.3. The identification $\mathrm{Pro}(\mathrm{FinSet}) \simeq \mathrm{ProfSet}$ restricts to $\mathrm{Pro}^{\aleph_1}(\mathrm{FinSet}) \simeq \mathrm{ProfSet}^{\mathrm{light}}$, i.e. $\mathrm{ProfSet}^{\mathrm{light}}$ is the category obtained from FinSet by freely adjoining the $\aleph_1$-small cofiltered limits. This is due to the fact that any countable filtered category admits a cofinal morphism from $\mathbb{N}$.[(1)]

Definition 1.4. The surjections in $\mathrm{ProfSet}^{\mathrm{light}}$ induce a canonical Grothendieck topology [Shv, Proposition 7]. The category of *light condensed anima* is the category of hypersheaves (of anima) on $\mathrm{ProfSet}^{\mathrm{light}}$ w.r.t. this Grothendieck topology, i.e. $\mathrm{CondAn}^{\mathrm{light}} = \mathbf{Sh}(\mathrm{ProfSet}^{\mathrm{light}})^{\wedge}$.

Remark 1.5. In particular, $\mathrm{CondAn}^{\mathrm{light}}$ is a hypercomplete ∞-topos. This the very reason for working with *light* profinite sets as that guarantees a small site of definition.

[(1)] In particular, categorical statements about countable cofiltered limits in general and cosequential limits in particular are formally equivalent and we will tacitly make use of this forthcoming.

Remark 1.6. Thus, a light condensed anima is a functor $X\colon (\mathrm{ProfSet}^{\mathrm{light}})^{op} \to \mathrm{An}$ that preserves finite products and satisfies descent for S-hypercovers [Shv, Propositions 2.1 & 2.3]. In [Cla25, Definition 4.6], Clausen spells this out and cites [Lur18b, Proposition A.5.7.2] as a justification, which however gives the characterization in terms of *semi-simplicial S-hypercovers* instead. I believe this to be an accident, and have only provided [Shv, Proposition 2.3] for completeness' sake. The precise descent condition does not actually matter in the sequel.

Proposition 1.7. *The Grothendieck topology on* $\mathrm{ProfSet}^{\mathrm{light}}$ *is subcanonical, i.e. the Yoneda embedding*[(2)] *factors as* $\mathrm{ProfSet}^{\mathrm{light}} \hookrightarrow \mathrm{CondAn}^{\mathrm{light}}$.

Proof. Any representable presheaf preserves small colimits, in particular finite coproducts. Furthermore, if $S_\bullet \to S$ is a hypercover, then $S_0 \to S$ is a surjection between compact Hausdorff spaces and thus a quotient map. It follows that $S_0 \times_S S_0 \rightrightarrows S_0 \to S$ and hence $S_1 \rightrightarrows S_0 \to S$ is a coequalizer in $\mathrm{ProfSet}^{\mathrm{light}}$, so mapping out yields the necessary equalizer (this computes the limit of the Čech nerve in the 1-category $\mathrm{Set} \subseteq \mathrm{An}$). □

Proposition 1.8. *The ∞-topos* $\mathrm{CondAn}^{\mathrm{light}}$ *is* replete*, i.e. for any tower $\ldots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ with surjective transition maps in* $\mathrm{CondAn}^{\mathrm{light}}$*, the projection $\lim_n X_n \twoheadrightarrow X_0$ is a surjection.*

Proof. This reduces by the criterion of [Shv, Proposition 1.6] to the analogous claim about towers of surjections in $\mathrm{ProfSet}^{\mathrm{light}}$, which is elementary. □

Remark 1.9. Together with hypercompleteness, this implies that $\mathrm{CondAn}^{\mathrm{light}}$ is Postnikov-complete, see [Cla25, Lemma 4.10], but we will not use this.

Though I will not attempt to give a general philosophical motivation on (light) condensed anima, the next two definitions illustrate how to conceptualize them.

Definition 1.10 (Constant)**.** There is a canonical functor $\delta\colon \mathrm{An} \to \mathrm{CondAn}^{\mathrm{light}}$ taking an anima X to the constant hypersheaf (i.e. the hypersheafification of the constant presheaf) with value X. This is the unique such functor in LTop. A light condensed anima is called *constant* if it is contained in the essential image of δ.

Definition 1.11 (Static)**.** A light condensed anima is called *static* if it is a pointwise discrete anima.[(3)] There is a functor $\mathrm{Top} \to \mathrm{CondAn}^{\mathrm{light}}$, $X \mapsto \underline{X}$ given by $\underline{X}(S) = \mathrm{Hom}(S, X)$ that associates a static light condensed anima to any topological space.

Remark 1.12. The argument of Proposition 1.7 verbatim shows that the functor $\mathrm{Top} \to \mathrm{CondAn}^{\mathrm{light}}$ is well-defined. It also evidently preserves small limits, since $\mathrm{Set} \subseteq \mathrm{An}$ is stable under limits.

Thus, (light) condensed anima are objects having two distinct components: a homotopy-theoretic component captured by the values in anima, and a topological component captured by the site of definition. The static and constant light condensed anima are orthogonal examples of light condensed anima in the sense that they are 'discrete' in one sense and not in the other. It will be shown later that the functor δ is fully faithful, and it is an exercise in point-set topology that the functor $\underline{(-)}$ is fully faithful on e.g. Δ-generated or sequential spaces.

(2) This will henceforth be left notationally implicit.

(3) Equivalently, as in the proof of [Shv, Proposition 1.6], these are the 0-truncated light condensed anima, i.e. the sheaves of sets on $\mathrm{ProfSet}^{\mathrm{light}}$, which is why they may also be called light condensed sets.

2 Étale Sheaves

From ordinary sheaf theory, recall the following. For a topological space X, any map $f\colon Y \to X$ defines sheaf of sections $\Gamma(-, f)$. In the converse direction, any presheaf $\mathcal{F}$ on X induces a topological space called the *étalé space* $X_{\mathcal{F}} \to X$, which is the disjoint union of the stalks of $\mathcal{F}$ equipped with final topology with respect to all the sections of $\mathcal{F}$. These constructions determine an adjoint equivalence between $\mathbf{Sh}(X)$ and the category of local homeomorphisms, a.k.a. *étale maps*[4], with target X.

Goal: We want to use light condensed anima to construct étalé spaces for sheaves of anima (instead of sets) on a light profinite set S. Then, we want to bootstrap from this case to *define* 'étale sheaves' on a light condensed anima in terms of their étalé spaces and observe that this gives a well-behaved theory.

Definition 2.1. A map of light profinite sets $f\colon S \to T$ is *finite étale* if there is a map of finite sets $f'\colon S' \to T'$ and a cartesian square

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow & \lrcorner & \downarrow \\ S' & \xrightarrow{f'} & T'. \end{array}$$

Remark 2.2. It is an exercise in point-set topology to check that the finite étale maps between light profinite sets are precisely the local homeomorphisms.

Proposition 2.3. *Let S be a light profinite set. There is an essentially unique functor*

$$\delta_S\colon \mathbf{Sh}(S) \to \mathrm{CondAn}^{\mathrm{light}}_{/S},$$

which satisfies:

1. *It is a morphism in* LTop*, i.e. it preserves small colimits and finite limits.*
2. *It is fully faithful.*
3. *Its essential image consists of those $X \to S$ such that*

$$\left(\mathrm{ProfSet}^{\mathrm{light}}_{/S}\right)^{op} \to \mathrm{An},\ (f\colon T \to S) \mapsto \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T, X)$$

   preserves countable filtered colimits.[5]

4. *It is left Kan extended from $\mathrm{COp}_S \to \mathrm{CondAn}^{\mathrm{light}}_{/S}$, $U \mapsto (U \to S)$.*

Proof. The functor δ_S will first be constructed as a sequence of composites

$$\mathbf{Sh}(S) \xrightarrow{\sim} \mathbf{Sh}(\mathrm{COp}_S) \xleftarrow{\sim} \mathbf{Sh}\left(\mathrm{fÉt}_{/S}\right) \xleftarrow{\sim} \mathbf{PSh}^{\sqcup, acc^{\aleph_1}}\left(\mathrm{ProfSet}^{\mathrm{light}}_{/S}\right) \hookrightarrow \mathbf{Sh}\left(\mathrm{ProfSet}^{\mathrm{light}}_{/S}\right)^{\wedge} \xrightarrow{\sim} \mathrm{CondAn}^{\mathrm{light}}_{/S}.$$

[4] The verb is étale. The noun is étalé.

[5] This (pre-)sheaf is precisely the one associated to $X \to S$ under the equivalence $\mathbf{Sh}(\mathrm{ProfSet}^{\mathrm{light}}_{/S})^{\wedge} \simeq \mathrm{CondAn}^{\mathrm{light}}_{/S}$.

Here, COp_S is the site of compact open subsets of S, equipped with the Grothendieck topology of finite open covers, and the inclusion $\mathrm{COp}_S \hookrightarrow \mathrm{Op}_S$ satisfies the hypotheses of [Hoy14, Lemma C.3] since the compact open subsets form a basis of the topology of S, hence $\mathbf{Sh}(S) \xrightarrow{\sim} \mathbf{Sh}(\mathrm{COp}_S)$ is an equivalence.

The category of $\mathrm{f\acute{E}t}_{/S}$ of (finite) étale maps $T \to S$ and the class of surjections satisfy the hypotheses of [Shv, Proposition 2.1], hence determine a site $\mathrm{f\acute{E}t}_{/S}$ and the inclusion $\mathrm{COp}_S \hookrightarrow \mathrm{f\acute{E}t}_S$ also satisfies the hypotheses of [Hoy14, Lemma C.3], hence $\mathbf{Sh}(\mathrm{f\acute{E}t}_{/S}) \xrightarrow{\sim} \mathbf{Sh}(\mathrm{COp}_S)$ is an equivalence. Indeed, this is a consequence of the observation that, for a finite étale map $f\colon T \to S$, any point in the domain lies in the image of of a section defined over the compact open subspace $f(T)$ (as the same is trivially true for any map of finite sets). Furthermore, a presheaf $\mathcal{F}\colon \mathrm{f\acute{E}t}_{/S}^{op} \to \mathrm{An}$ is a sheaf if and only if it preserves finite products. This is a consequence of [Shv, Proposition 2.1] since any finite étale surjection f admits a section and thus $\mathcal{F}\check{C}(f)$ is a limit diagram for purely diagrammatic reasons [Lur18a, Tag 04SH].

To wit the next equivalence, note that $\mathrm{Pro}^{\aleph_1}(\mathrm{f\acute{E}t}_{/S}) \simeq \mathrm{ProfSet}^{\mathrm{light}}_{/S}$ (this is an exercise in abstract nonsense), so restriction yields an equivalence $\mathbf{PSh}^{\mathrm{acc}^{\aleph_1}}(\mathrm{ProfSet}^{\mathrm{light}}_{/S}) \xrightarrow{\sim} \mathbf{PSh}(\mathrm{f\acute{E}t}_{/S})$, where the superscript indicates '$\aleph_1$-small filtered colimit-preserving'. This restricts to an equivalence between the full subcategories of finite product-preserving functors on either side, which is to say that a countable filtered colimit-preserving functor $(\mathrm{ProfSet}^{\mathrm{light}}_{/S})^{op} \to \mathrm{An}$ preserves finite products if and only its restriction $\mathrm{f\acute{E}t}_{/S}^{op} \to \mathrm{An}$ preserves finite products. This holds since any object of $(\mathrm{ProfSet}^{\mathrm{light}}_{/S})^{op}$ is a countable filtered colimit of objects in $\mathrm{f\acute{E}t}_{/S}^{op}$ and filtered colimits and finite products in An commute.

The next inclusion claims that any presheaf $\mathcal{F}\colon (\mathrm{ProfSet}^{\mathrm{light}}_{/S})^{op} \to \mathrm{An}$ that preserves finite products and countable filtered colimits is already a hypersheaf. To this end, note that $\mathbf{PSh}^{\sqcup,\mathrm{acc}^{\aleph_1}}(\mathrm{ProfSet}^{\mathrm{light}}_{/S})$ is stable under pointwise Postnikov-truncation as the latter preserves finite products and small colimits, so any such presheaf is the cosequential limit of truncated such presheaves. Since $\mathbf{Sh}(\mathrm{ProfSet}^{\mathrm{light}}_{/S})$ is stable under limits, we may assume w.l.o.g. that $\mathcal{F}$ factors through $\tau_{\leq n}\mathrm{An} \subseteq \mathrm{An}$. The truncated objects in an ∞-topos are always hypercomplete [Lur09, Lemma 6.5.2.9], so it suffices to prove that $\mathcal{F}$ is a sheaf, i.e. by [Shv, Proposition 2.1] that it takes Čech nerves of surjections to limit diagrams. To this end, let $f\colon X \twoheadrightarrow Y$ be a surjection in $\mathrm{ProfSet}^{\mathrm{light}}_{/S}$ and consider the diagram

$$\Delta^{inj}_{\leq n+1,+} \hookrightarrow \Delta_+ \xrightarrow{\check{C}(f)} \left(\mathrm{ProfSet}^{\mathrm{light}}_{/S}\right)^{op} \xrightarrow{\mathcal{F}} \tau_{\leq n}\mathrm{An} \xrightarrow{\iota} \mathrm{An}.$$

The subcategory $\tau_{\leq n}\mathrm{An} \subseteq \mathrm{An}$ is an $(n+1)$-category stable under limits and the functor $\Delta^{inj}_{\leq n+1} \hookrightarrow \Delta$ is $(n+1)$-limit cofinal by [Du23, Corollary 6.13 & Example 6.14]. Thus, $\iota\mathcal{F}\check{C}(f)$ is a limit diagram if and only if $\mathcal{F}\check{C}(f)|_{\Delta^{inj}_{\leq n+1,+}}$ is a limit diagram and the upshot is that $\Delta^{inj}_{\leq n+1}$ is a *finite* category. To exploit this, note that f can be written as a cosequential limit of finite étale surjections $f_n\colon X_n \twoheadrightarrow Y_n$ in $\mathrm{ProfSet}^{\mathrm{light}}_{/S}$ (another exercise). The assumption on $\mathcal{F}$ then implies that $\mathcal{F}\check{C}(f) \simeq \mathrm{colim}_n \mathcal{F}\check{C}(f_n)$ and filtered colimits commute with finite limits in An, so finite limit diagrams are stable under filtered colimits and we are reduced to the case where f is a finite étale surjection. This case was already covered two paragraphs prior.

The last equivalence is an instance of [Shv, Corollary 3.8], so we have finally constructed δ_S and properties 2. and 3. are immediate consequences of the construction. To prove 1., note that the essential image described

in 3. is clearly stable under finite limits (since filtered colimits commute with finite limits in An), so it remains to prove that the essential image is stable under small colimits.[6] To this end, it suffices to separately prove that it is stable under a) small sifted colimits and b) finite coproducts.[7]

To prove a), note that $\mathbf{PSh}^{\sqcup,acc^{\aleph_1}}(\mathrm{ProfSet}^{\mathrm{light}}_{/S})$ is stable under small sifted colimits in $\mathbf{PSh}(\mathrm{ProfSet}^{\mathrm{light}}_{/S})$ (since they commute with finite products and countable filtered colimits) and that these colimits are thus *a fortiori* colimits in $\mathbf{Sh}(\mathrm{ProfSet}^{\mathrm{light}}_{/S})^{\wedge}$ due to the previous argument that a finite product- and countable filtered colimit-preserving presheaf is automatically a hypersheaf.

To prove b), it suffices to check the initial object and binary coproducts. The essential image contains all finite étale maps $T \to S$ of light profinite sets, so in particular the initial object $\emptyset \to S$. To handle binary coproducts, assume that $X, Y \in \mathrm{CondAn}^{\mathrm{light}}_{/S}$ are contained in the essential image, let $T = \lim_n T_n$ be a cosequential limit in $\mathrm{ProfSet}^{\mathrm{light}}_{/S}$ and consider the commutative square

$$\begin{array}{ccc} \mathrm{colim}_n \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T_n, X \sqcup Y) & \xrightarrow{\alpha} & \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T, X \sqcup Y) \\ \downarrow & & \downarrow \\ \mathrm{colim}_n \mathrm{Map}_{\mathrm{ProfSet}^{\mathrm{light}}}(T_n, \{0,1\}) & \xrightarrow{\sim} & \mathrm{Map}_{\mathrm{ProfSet}^{\mathrm{light}}}(T, \{0,1\}) \end{array}$$

induced by the map $X \sqcup Y \to \{0,1\}$. The bottom horizontal map is an equivalence since $\{0,1\}$ is a finite set. A map $T \to \{0,1\}$ corresponds to a decomposition $T \cong T_0 \sqcup T_1$ and is induced by compatible decompositions $T_n \cong T_{n,0} \sqcup T_{n,1}$ for $n \gg 0$. Then, a small calculation using the disjointness of coproducts in $\mathrm{CondAn}^{\mathrm{light}}$ identifies the map induced by α on fibers over this decomposition as

$$\operatorname*{colim}_n \left(\mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T_{n,0}, X) \times \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T_{n,1}, Y) \right) \to \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T_0, X) \times \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(T_1, Y)$$

and this is an equivalence due to the assumption on X, Y, and since cosequential limits commute with finite products. Thus, the map α is an equivalence, i.e. $X \sqcup Y$ is contained in the essential image of δ_S.

In conclusion, property 1. is established. Lastly, since δ_S preserves small colimits and the Yoneda embedding $\mathrm{COp}_S \hookrightarrow \mathbf{Sh}(\mathrm{COp}_S) \simeq \mathbf{Sh}(S)$ is dense, property 4. is immediate as soon as we note that $\delta_S|_{\mathrm{COp}_S}$ identifies with the claimed functor. This holds by squinting one's eyes till conviction is reached. □

Remark 2.4. The functoriality of this construction is not properly discussed in [Cla25, Section 4], so let me dwell on it for a bit. Let $f\colon S \to T$ be a map of light profinite sets. There is a commutative diagram of sites

$$\begin{array}{ccccc} \mathrm{Op}_T & \longleftarrow & \mathrm{COp}_T & \longrightarrow & \mathrm{fÉt}_{/T} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Op}_S & \longleftarrow & \mathrm{COp}_S & \longrightarrow & \mathrm{fÉt}_{/S}, \end{array}$$

where the vertical morphisms are pullbacks along f. This gives functoriality of the first two equivalences with respect to the *pushforward* of sheaves along f. In the third equivalence, this corresponds to the pullback along

[6] I would like to thank Kaif Hilman for pointing out a critical error in a previous version of this argument.
[7] This can be deduced from the Bousfield-Kan formula.

$f^*\colon \mathrm{ProfSet}^{\mathrm{light}}_{/T} \to \mathrm{ProfSet}^{\mathrm{light}}_{/S}$ and this is again a morphism of sites, hence we obtain

$$\begin{array}{ccccc} \mathbf{Sh}(S) & \xhookrightarrow{\delta_S} & \mathbf{Sh}\left(\mathrm{ProfSet}^{\mathrm{light}}_{/S}\right)^{\wedge} & \xrightarrow{\sim} & \mathrm{CondAn}^{\mathrm{light}}_{/S} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (f^*)^\circ} & & \downarrow{\scriptstyle f_*} \\ \mathbf{Sh}(T) & \xhookrightarrow{\delta_T} & \mathbf{Sh}\left(\mathrm{ProfSet}^{\mathrm{light}}_{/T}\right)^{\wedge} & \xrightarrow{\sim} & \mathrm{CondAn}^{\mathrm{light}}_{/T}. \end{array} \tag{1}$$

The commutativity of the right-hand square can be deduced from the discussion in [Shv, Remark 3.6] since f^* is right-adjoint to the post-composition $f_!\colon \mathrm{ProfSet}^{\mathrm{light}}_{/S} \to \mathrm{ProfSet}^{\mathrm{light}}_{/T}$ and thus pre-composing with it computes the right Kan extension along $f_!$.

Remark 2.5. Furthermore, the total square in (1) is vertically left-adjointable. To see this, since the horizontal maps are fully faithful, it suffices to observe that the left adjoint $f^*\colon \mathrm{CondAn}^{\mathrm{light}}_{/T} \to \mathrm{CondAn}^{\mathrm{light}}_{/S}$ maps the essential image of δ_T into that of δ_S, which can be deduced from 3. of the proposition. Indeed, letting $X \to T$ belong to the essential image of δ_T and computing

$$\mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(S', f^*X) \simeq \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/T}}(f_!S', X), \qquad \forall (S' \to S) \in \mathrm{ProfSet}^{\mathrm{light}}_{/S},$$

this is immediate from $f_!\colon \mathrm{ProfSet}^{\mathrm{light}}_{/S} \to \mathrm{ProfSet}^{\mathrm{light}}_{/T}$ preserving (countable) cofiltered limits (as these are computed underlying). In particular, applying this observation to a sheaf $\mathcal{F} \in \mathbf{Sh}(T)$ and a point $t\colon * \to T$, we obtain a cartesian square

$$\begin{array}{ccc} \delta(\mathcal{F}_t) & \longrightarrow & \delta_T(\mathcal{F}) \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & T. \end{array}$$

In other words, the fiber of the étale map $\delta_T(\mathcal{F}) \to T$ that classifies the sheaf $\mathcal{F}$ over a point $t \in T$ is the constant light condensed anima corresponding to the stalk $\mathcal{F}_t$ of $\mathcal{F}$ at t. This is as it should be.

Theorem 2.6. *Let $f\colon X \to Y$ be a map of light condensed anima. The following conditions are equivalent:*

1) For every cosequential limit $S = \lim_n S_n$ in $\mathrm{ProfSet}^{\mathrm{light}}$, the following square is cartesian:

$$\begin{array}{ccc} \mathrm{colim}_n X(S_n) & \xrightarrow{f} & \mathrm{colim}_n Y(S_n) \\ \downarrow & \lrcorner & \downarrow \\ X(S) & \xrightarrow{f} & Y(S). \end{array}$$

2) The functor $(\mathrm{ProfSet}^{\mathrm{light}})^{op}_{/Y} \to \mathrm{An}$, $(S \to Y) \mapsto \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/Y}}(S, X)$ preserves countable filtered colimits.

3) For every $S \in \mathrm{ProfSet}^{\mathrm{light}}$ and $S \to Y$, the pullback $X \times_Y S \to S$ belongs to the essential image of δ_S.

The maps satisfying these equivalent conditions are called the étale maps.

Proof. Fix a cosequential limit $S = \lim_n S_n$ in $\mathrm{ProfSet}^{\mathrm{light}}$. The points of $\mathrm{colim}_n Y(S_n)$ are points of $Y(S_{n_0})$ for $n_0 \gg 0$ and these correspond to maps $S_{n_0} \to Y$ for $n_0 \gg 0$. Fixing one such point, consider the identifications

$$\begin{array}{ccc}
\mathrm{colim}_{n\geq n_0} \mathrm{fib}_{S_n \to S_{n_0} \to Y}(X(S_n) \to Y(S_n)) & \longrightarrow & \mathrm{fib}_{S \to S_{n_0} \to Y}(X(S) \to Y(S)) \\
\wr & & \wr \\
\mathrm{colim}_{n\geq n_0} \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/Y}}(S_n, X) & \longrightarrow & \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/Y}}(S, X) \\
\wr & & \wr \\
\mathrm{colim}_{n\geq n_0} \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S_{n_0}}}(S_n, X \times_Y S_{n_0}) & \longrightarrow & \mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S_{n_0}}}(S, X \times_Y S_{n_0}).
\end{array}$$

The condition 1) is equivalent to the upper horizontal map in this diagram always being an equivalence, since pullbacks can be checked on horizontal fibers and filtered colimits commute with fibers. The condition 2) is definitionally equivalent to the middle horizontal map in this diagram always being an equivalence, and the condition 3) is equivalent to the bottom horizontal map in this diagram always being an equivalence by Proposition 2.3 and unwinding the quantifiers. □

Remark 2.7. In the case $Y = S$ is a light profinite set, characterization 2) of Theorem 2.6 and 3. of Proposition 2.3 imply that $f\colon X \to S$ is étale if and only if it is contained in the essential image of δ_S.

Example 2.8 (Constant). Taking $S = *$, the morphism $\delta\colon \mathrm{An} \simeq \mathbf{Sh}(*) \hookrightarrow \mathrm{CondAn}^{\mathrm{light}}$ is necessarily the 'constant' functor of Definition 1.10, so in particular this functor is fully faithful and the constant light condensed anima are precisely those that are étale over the point. Furthermore, the proposition tells us how to compute $\delta(X)$: for a light profinite set $S = \lim_n S_n$ written as a cosequential limit of finite sets, we have $\delta(X)(S) = \mathrm{colim}_n \delta(X)(S_n) = \mathrm{colim}_n \prod_{S_n} X$. In particular, we see that if X is finite and discrete, then $\delta(X)$ is the same as X interpreted as a light profinite set.

Corollary 2.9. *1. The collection of étale morphisms contains the equivalences and is stable under coproducts and pullbacks. Furthermore, given maps of light condensed anima $f\colon X \to Y$ and $g\colon Y \to Z$, if g is étale, then f is étale if and only if gf is étale.*

2. The full subcategory of $\mathrm{CondAn}^{\mathrm{light}}_{/Y}$ spanned by the étale maps is stable under small colimits and finite limits.

Proof. The stability under pullbacks in 1. follows by characterization 3) and the other properties by characterization 1) of Theorem 2.6. For 2., this follows from characterization 3) of Theorem 2.6 and 1. of Proposition 2.3 (as well as universality of colimits in $\mathrm{CondAn}^{\mathrm{light}}$). □

To avoid going into technical details about Postnikov completions, we shall state and use the following very useful property without proof.

Proposition 2.10 ([Cla25, Lemma 4.17])**.** *The collection of étale maps is 'local', i.e. if $f\colon X \to Y$ is a map of light condensed anima and $\pi\colon Y' \twoheadrightarrow Y$ is a surjection of light condensed anima such that the pullback $\pi^* f\colon X \times_Y Y' \to Y'$ is étale, then f is étale.* □

For no good reason, we denote $\mathbf{Sh}_v(Y) \coloneqq \mathrm{CondAn}^{\mathrm{light}}_{/Y}$ and call these *v-sheaves.*[8] Now, by the bulk of [Lur09, Sections 6.1-6.3], taking v-sheaves with the pullback functoriality defines a limit-preserving functor

$$\mathbf{Sh}_v(-) : \left(\mathrm{CondAn}^{\mathrm{light}}\right)^{op} \to \mathrm{LTop}.$$

The essence of étale sheaves 'behaving like ordinary sheaves' is captured by the following proposition.

Proposition 2.11. *Taking étale sheaves with the pullback functoriality defines a limit-preserving functor*

$$\mathbf{Sh}_{\acute{e}t}(-) : \left(\mathrm{CondAn}^{\mathrm{light}}\right)^{op} \to \mathrm{LTop}.$$

Proof. The pullback-stability of étale maps implies that $\mathbf{Sh}_{\acute{e}t}(-) : \mathrm{CondAn}^{\mathrm{light}} \to \widehat{\mathrm{Cat}}_\infty$ exists as a subfunctor of $\mathbf{Sh}_v(-) : \mathrm{CondAn}^{\mathrm{light}} \to \mathrm{LTop} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$. The inclusion $\mathrm{LTop} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ preserves small limits [Lur09, Proposition 6.3.2.3] and fully faithful functors are stable under limits, so it makes sense to claim that, for any diagram $X_\bullet : \mathcal{I} \to \mathrm{CondAn}^{\mathrm{light}}$, the equivalence $\mathbf{Sh}_v(\mathrm{colim}_{i\in\mathcal{I}} X_i) \simeq \lim_{i\in\mathcal{I}} \mathbf{Sh}_v(X_i)$ restricts to an equivalence $\mathbf{Sh}_{\acute{e}t}(\mathrm{colim}_{i\in\mathcal{I}} X_i) \simeq \lim_{i\in\mathcal{I}} \mathbf{Sh}_{\acute{e}t}(X_i)$, i.e. a map $Y \to \mathrm{colim}_{i\in\mathcal{I}} X_i$ is étale if and only if the pullback $Y \times_{\mathrm{colim}_{i\in\mathcal{I}} X_i} X_i \to X_i$ is étale for all $i \in I$. This is indeed the case as a consequence of locality (and universality of coproducts) since $\bigsqcup_{i\in\mathcal{I}} X_i \twoheadrightarrow \mathrm{colim}_{i\in\mathcal{I}} X_i$ is a surjection by [Lur09, Lemma 6.2.3.13].

Thus, we have obtained a limit-preserving functor $\mathbf{Sh}_{\acute{e}t}(-) : \mathrm{CondAn}^{\mathrm{light}} \to \widehat{\mathrm{Cat}}_\infty$ and we need to check that it factors through $\mathrm{LTop} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$. To see this[9], note that its restriction to $\mathrm{ProfSet}^{\mathrm{light}}$ along the Yoneda embedding factors through $\mathrm{LTop} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ by Proposition 2.3 & Remark 2.4 (indeed, for any $f : S \to T$, the pullback $f^* : \mathbf{Sh}(T) \to \mathbf{Sh}(S)$ is a morphism in LTop). The claim then follows since the Yoneda embedding is dense, $\mathbf{Sh}_{\acute{e}t}(-)$ is limit-preserving and $\mathrm{LTop} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ is stable under limits. □

Corollary 2.12. *The functor* $\mathbf{Sh}_{\acute{e}t}(-) : (\mathrm{CondAn}^{\mathrm{light}})^{op} \to \mathrm{LTop}$ *is right Kan extended along the Yoneda embedding from* $(\mathrm{ProfSet}^{\mathrm{light}})^{op} \to \mathrm{LTop}$, $S \mapsto \mathbf{Sh}(S)$. □

Remark 2.13. For any light profinite set S, the ∞-topos $\mathbf{Sh}(S) \simeq \mathbf{Sh}(\mathrm{f\acute{E}t}_{/S})$ is Postnikov-complete, because the sheaf condition in the latter is simply the preservation of finite products (cf. proof of Proposition 2.3), so that Postnikov-truncation can be computed sectionwise (as Postnikov-truncation commutes with finite products). The same argument, using that Postnikov-complete ∞-topoi are closed under limits in LTop (e.g. [Hai22, 1.25]), then shows that $\mathbf{Sh}_{\acute{e}t}(X)$ is a Postnikov-complete ∞-topos for all $X \in \mathrm{CondAn}^{\mathrm{light}}$.

Moreover, the 'points' $* \to X$ induce morphisms $\mathbf{Sh}_{\acute{e}t}(X) \to \mathbf{Sh}_{\acute{e}t}(*) \simeq \mathrm{An}$ in LTop that are jointly conservative, i.e. $\mathbf{Sh}_{\acute{e}t}(X)$ has enough points. To see this, we can reduce to the case where $S \in \mathrm{ProfSet}^{\mathrm{light}}$ by pulling back along maps from representables and then $\mathbf{Sh}(S)$ has enough points, e.g. by combining the previous observation with the argument from [Lur09, Corollary 7.2.1.17].

[8] The 'v' stands for 'valuation' and originates from work of Scholze. It really makes no sense in this context.
[9] I would like to thank Maxime Ramzi for pointing out this argument during the talk.

3 Examples

Finally, we can discuss a couple of examples.

Example 3.1. For an anima X, we have that $\delta(X) \to *$ is étale, so that a map $Y \to \delta(X)$ is étale if and only if $Y \to *$ is étale if and only if Y is also a constant light condensed anima. Since δ is fully faithful, it follows that $\mathbf{Sh}_{ét}(\delta(X)) \simeq \mathrm{An}_{/X} \simeq \mathrm{Fun}(X, \mathrm{An})$, i.e. étale sheaves on $\delta(X)$ are precisely the local systems of anima on X.

Proposition 3.2. *The functor* $\underline{(-)}\colon \mathrm{Top} \to \mathrm{CondAn}^{\mathrm{light}}$ *takes local homeomorphisms to étale maps.*

Proof. This is a (slightly challenging) exercise in point-set topology. □

Remark 3.3. Even better, we can obtain a commutative square

$$\begin{array}{ccc} \mathbf{Sh}(S;\mathrm{Set}) & \hookrightarrow & \mathbf{Sh}(S) \\ \downarrow & & \downarrow{\scriptstyle \delta_S} \\ \mathrm{Top}_{/S} & \xrightarrow{\underline{(-)}} & \mathrm{CondAn}^{\mathrm{light}}_{/S}, \end{array}$$

in which the left vertical map is the classical étalé space construction (i.e. our étalé spaces are a faithful extension of the classical concept). To prove this, simply note that for any $\mathcal{F} \in \mathbf{Sh}(S;\mathrm{Set})$ and corresponding étalé space $\pi\colon X_{\mathcal{F}} \to S$, we have natural identifications $\mathrm{Map}_{\mathrm{CondAn}^{\mathrm{light}}_{/S}}(U, \underline{X_{\mathcal{F}}}) \simeq \Gamma(U, \pi) \cong \mathcal{F}(U)$ for any compact open subset $U \subseteq S$. This implies that $\underline{\pi}\colon \underline{X_{\mathcal{F}}} \to S$ classifies $\mathcal{F}$.

The functor $\mathrm{Op}_X \to \mathbf{Sh}_{ét}(\underline{X}), U \mapsto (\underline{U} \to \underline{X})$ preserves finite limits and takes open covers to jointly surjective families by an application of [Shv, Proposition 1.6], hence extends to functor $\mathbf{Sh}(X) \to \mathbf{Sh}_{ét}(\underline{X})$ in LTop by [Lur09, Proposition 6.2.3.20]. The following Proposition, which we will not prove here, tells us that this functor is not too far from an equivalence under mild conditions.

Proposition 3.4 ([Cla25, Lemma 4.20, 4.])**.** *For a topological space X in which every point has a second-countable compact Hausdorff neighborhood, the functor $\mathbf{Sh}(X) \to \mathbf{Sh}_{ét}(\underline{X})$ exhibits the target as a Postnikov-completion of the source.* □

Example 3.5. First of all, we recall from higher topos theory that, in any ∞-topos $\mathcal{X}$,

- the Čech nerve and geometric realization define inverse equivalences between the ∞-category of groupoid objects in $\mathcal{X}$ and the ∞-category of surjections in $\mathcal{X}$ [Lur09, Corollary 6.2.3.5], and
- a group object G in $\mathcal{X}$ determines a surjection $* \to BG$ under this equivalence and pullback induces an equivalence between the slice $\mathcal{X}_{/BG}$ and the ∞-category of objects of $\mathcal{X}$ equipped with a group action of G internal to $\mathcal{X}$ (e.g. [Hoy26, Section 6.2]).

Now, let $G \cong \lim_n G_n$ be a light profinite group written as a cosequential limit of finite groups with surjective transition maps. From the first fact, we obtain that $(* \twoheadrightarrow BG) \simeq \lim_n (* \twoheadrightarrow BG_n)$ in the category of surjections in $\mathrm{CondAn}^{\mathrm{light}}$. The map $* \to \lim_n BG_n$, however, is surjective (this will be proven below), so this limit is a posteriori computed in the category of arrows in $\mathrm{CondAn}^{\mathrm{light}}$, in particular $BG \simeq \lim_n BG_n$. Since G_n is a

finite group and δ is a morphism in LTop, we have that $BG_n \simeq \delta(BG_n)$ is the constant light condensed anima associated to the usual classifying space of G_n.

From the second fact, we obtain that $\mathbf{Sh}_v(BG) \simeq \mathrm{CondAn}^{\mathrm{light}}_{/BG}$ is the category of light condensed anima equipped with a G-action and $\mathbf{Sh}_{\acute{e}t}(BG)$ is the full subcategory on the *constant* light condensed anima equipped with a G-action (since being étale can be tested after pullback along the surjection $* \twoheadrightarrow BG$).

To verify the surjectivity claim, note that, for any $n \geq 0$, the map $(\lim_n BG_n) \times_{BG_{n+1}} * \to (\lim_n BG_n) \times_{BG_n} *$ is a pullback of $* \to BG_{n+1} \times_{BG_n} *$, which identifies with the surjection $* \twoheadrightarrow B\ker(G_{n+1} \twoheadrightarrow G_n)$. Thus, we have a tower of surjections and repleteness implies that the limit $* \twoheadrightarrow \lim_n BG_n$ is surjective, as desired.

To conclude, we combine these with concepts from the first talk to introduce the geometric objects on which we will later construct a six-functor formalism.

Definition 3.6. Let F be a local field. The category Man_F of F-manifolds becomes a (large) site with the Grothendieck pretopology given by open covers.

Remark 3.7. The category $\mathbf{Sh}(\mathrm{Man}_F)$ of sheaves of F-manifolds is an ∞-topos, because we may argue (using [Hoy14, Lemma C.3]) that it is equivalent to sheaves on the small site of open subspaces of F^d, $d \geq 0$.

Proposition 3.8. *Let F be a local field. There is a unique colimit-preserving functor $\mathbf{Sh}(\mathrm{Man}_F) \to \mathrm{CondAn}^{\mathrm{light}}$ extending $M \mapsto \underline{M^{top}}$. Furthermore, it preserves pullbacks along maps that can be locally represented as submersions with respect to the Grothendieck topology.*

Proof. The functor $\mathrm{Man}_F \to \mathrm{CondAn}^{\mathrm{light}}$, $M \mapsto \underline{M^{top}}$ preserves pullbacks along submersions and takes open covers to jointly surjective families by an application of [Shv, Proposition 1.6], so this holds by an argument along the lines of [Lur09, Proposition 6.2.3.20].[(10)] □

Definition 3.9. Let F be a local field. An *F-analytic smooth Artin stack* is a sheaf $X \in \mathbf{Sh}(\mathrm{Man}_F)$ for which there exists an $M \in \mathrm{Man}_F$ and a surjective representable submersion $M \twoheadrightarrow X$, i.e. for any $N \in \mathrm{Man}_F$ and map $N \to X$, the pullback $M \times_X N \to N$ is representable by a submersion in Man_F.

Example 3.10. Let F be a local field, G an F-Lie group and M an F-manifold equipped with an action of G (i.e. internally in Man_F). Then $\pi\colon M \twoheadrightarrow M /\!\!/ G$ is an F-analytic smooth Artin stack. This map is the geometric realization of the action groupoid, hence a surjection by construction. To see that it is a representable submersion, consider any $M' \in \mathrm{Man}_F$ and map $M' \to M /\!\!/ G$. By [Shv, Proposition 1.6], there is an open cover $(U_i \to M')_{i \in I}$ that lifts through $\pi\colon M \to M /\!\!/ G$. Then, consider the diagrams

$$\begin{array}{ccccc} U_i \times_{M/\!\!/G} M & \longrightarrow & M' \times_{M/\!\!/G} M & \longrightarrow & M \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U_i & \longrightarrow & M' & \longrightarrow & M /\!\!/ G. \end{array}$$

[(10)] The only subtlety here is that the category Man_F does not have all pullbacks.

Using that $U_i \to M' \to M /\!\!/ G$ lifts through $\pi\colon M \to M /\!\!/ G$, its composite identifies with the composite of

$$\begin{array}{ccccc} U_i \times G & \longrightarrow & M \times G & \longrightarrow & M \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U_i & \longrightarrow & M & \longrightarrow & M /\!\!/ G. \end{array}$$

Thus, the representability criterion of the first talk implies that $M' \times_{M/\!\!/G} M \to M'$ is representable and the above description also witnesses that it is a submersion (it is locally a projection).

Note that we have worked in the topos $\mathbf{Sh}(\mathrm{Man}_F)$, but these geometric realizations and pullbacks are all preserved by the functor $\mathbf{Sh}(\mathrm{Man}_F) \to \mathrm{CondAn}^{\mathrm{light}}$ of Proposition 3.8, so that the entire picture translates into light condensed anima. Lastly, we remark that there is also a relative version over a base manifold $N \in \mathrm{Man}_F$, using the same construction in $\mathbf{Sh}((\mathrm{Man}_F)_{/N}) \simeq \mathbf{Sh}(\mathrm{Man}_F)_{/N} \to \mathrm{CondAn}^{\mathrm{light}}_{/\underline{N^{top}}}$.

References

[Cla25] Dustin Clausen. *Duality and linearization for p-adic lie groups*. 2025. arXiv: 2506.18174 [math.AT]. URL: https://arxiv.org/abs/2506.18174.

[Du23] Peng Du. *Reshaping limit diagrams and cofinality in higher category theory*. 2023. arXiv: 2308.01798 [math.CT]. URL: https://arxiv.org/abs/2308.01798.

[Hai22] Peter J. Haine. *Descent for sheaves on compact Hausdorff spaces*. 2022. arXiv: 2210.00186 [math.AT]. URL: https://arxiv.org/abs/2210.00186.

[Hoy14] Marc Hoyois. "A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula". In: *Algebraic & Geometric Topology* 14.6 (2014), pp. 3603–3658.

[Hoy26] Marc Hoyois. *Lectures on Higher Topos Theory*. Lecture Notes by Bastiaan Cnossen. 2026. URL: https://bastiaancnossen.com/notes/topos-theory.pdf.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009.

[Lur18a] Jacob Lurie. *Kerodon*. 2018. URL: https://kerodon.net.

[Lur18b] Jacob Lurie. *Spectral Algebraic Geometry*. 2018. URL: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[Shv] Thorger Geiß. *Notes on Sites and Sheaves*. 2026. URL: https://thorgergeiss.gitlab.io/Notes_on_Sites_and_Sheaves.pdf.