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Bordism-type Theories and (∞, ∞)-categories

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1 Introduction

1.1 Motivation

The central motif of algebraic topology is the study of spaces using algebraic invariants. Among these, the most accessible are the singular (co-)homology groups and their relatives, the generalized (reduced) (co-)homology theories, axiomatized by Eilenberg and Steenrod. These are given by a sequence of (contra-)variant functors $hCW_* \to \mathbf{Ab}$ on the homotopy category of pointed CW-complexes, satisfying additivity, exactness and a suspension axiom. For any cohomology theory $(h^n)_{n\in\mathbb{Z}}$, Brown's representability theorem implies that there are pointed spaces Z_n such that $h^n(X) \cong [X, Z_n]_*$ for all $n \in \mathbb{Z}$. The natural isomorphisms $h^n(X) \cong h^{n+1}(\Sigma X)$ correspond to homotopy equivalences $Z_n \xrightarrow{\sim} \Omega Z_{n+1}$, so that these spaces assemble into an (Ω-)*spectrum*. In the other direction, every spectrum defines a cohomology theory and, less obviously, a homology theory.

The appropriate setting to study (co-)homology theories and, more generally, stable phenomena is the category of spectra, which admits the smash product and a triangulated structure, so that the methods of homological algebra can be applied. These structures historically came with technical complications, which have only been resolved by instead using the stable ∞-category $\mathbf{Sp}$ of spectra. The classical and modern theory is given in [Ada74, Part III] and [Lur17, Chapter 1] respectively.

One of the most important geometrically flavored examples of (co-)homology theories are the theories of *(co-)bordism* introduced in [Ati61]. For a CW-complex X, the n-th bordism homology group is given by equivalence classes of maps $f\colon M \to X$ on n-dimensional compact, smooth manifolds M, where $(M, f) \sim (N, g)$ if and only if there exists an $(n+1)$-dimensional compact, smooth manifold with boundary $\partial W \cong M \sqcup N$ and a map $H\colon W \to X$ such that $H|_{\partial W}$ agrees with $f \sqcup g$. The spectrum governing (co-)bordism is the *Thom spectrum* MO. The algebraic analogues are Ranicki's *L-theory spectra* of [Ran92], which feature prominently in surgery theory.

Inspired by these examples, Quinn proposed the concept of 'bordism-type theories' in [Qui95], and, inspired by Quinn, Laures and McClure proposed the concept of 'ad theories' in [LM14]. These theories consist of algebraic data that, roughly speaking, provides objects graded by dimension and a collection of (higher) bordisms between them. The definitions, however, are ad hoc and insufficient insofar as they do not capture the additional structure that is usually present, e.g. *bordism categories* $\mathbf{Bord}_n$, consisting of $(n-1)$-dimensional objects and n-dimensional bordisms between them. In the case of ordinary bordism, these categories appear as the domains of *TQFTs* ([Ati88], [Lur09c]).

A modern algebraic model of a 'bordism-type theory' should thus be ∞-categorical and govern both a 'bordism spectrum' as well as the ordinary 'bordism categories'. To address this, we propose a

Philosophy. The algebraic models for 'bordism-type theories' are *categorical spectra*, a joint generalization of symmetric monoidal (∞,∞)-categories and ordinary spectra.

1.2 Results

The bulk of this thesis is concerned with implementing the ∞-categorical theory of categorical spectra, based on Barwick's iterated complete Segal spaces. This theory is not novel per se, yet our approach is different from the standard approach in the modern literature based on Higher Algebra, resulting in a partially novel account.

The ∞-categories $\mathbf{Cat}_{(\infty,n)}$ of (∞,n)-categories will be defined by iterating the construction of complete Segal space objects, starting with the ∞-category $\mathbf{Spc}$ of spaces. The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ of (∞,∞)-categories is then obtained as the limit of the diagram

$$\ldots \xrightarrow{\kappa_{n+1}} \mathbf{Cat}_{(\infty,n+1)} \xrightarrow{\kappa_n} \mathbf{Cat}_{(\infty,n)} \xrightarrow{\kappa_{n-1}} \ldots \xrightarrow{\kappa_0} \mathbf{Cat}_{(\infty,0)},$$

where the transfer maps are given by passing to the smallest appropriate subcategory. The embedding $\mathbf{Spc} \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$ admits both adjoints, and the left adjoint $|-| : \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Spc}$, called *geometric realization*, preserves finite products. Then, the ∞-category $\mathbf{CatSp}$ of categorical spectra is constructed according to the following table of analogies (compare [Ste21, p. 281]):

Algebraic Topology	Higher Category Theory
pointed spaces $\mathbf{Spc}_*$	pointed (∞,∞)-categories $\mathbf{Cat}_{(\infty,\infty),*}$
loop space $\Omega\colon \mathbf{Spc}_* \to \mathbf{Spc}_*$	loop object $\Omega\colon \mathbf{Cat}_{(\infty,\infty),*} \to \mathbf{Cat}_{(\infty,\infty),*}$
spectra $\mathbf{Sp}$	categorical spectra $\mathbf{CatSp}$
grouplike $\mathbb{E}_\infty$-spaces $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc})$	symmetrical monoidal (∞,∞)-categories $\mathbf{Cat}^{\otimes}_{(\infty,\infty)}$
adjunction $B^\infty\colon \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc}) \rightleftarrows \mathbf{Sp}\colon \Omega^\infty$	adjunction $B^\infty\colon \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \rightleftarrows \mathbf{CatSp}\colon \Omega^\infty$

The bottom row relates to the delooping hypothesis as well as May's recognition principle. The relationship of the two columns is explained by the

Theorem . *There is a horizontally left and vertically right adjointable square*

$$\begin{array}{ccc} \mathbf{CatSp} & \xrightarrow{\Omega^\infty} & \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \\ \uparrow & & \uparrow \\ \mathbf{Sp} & \xrightarrow{\Omega^\infty} & \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc}). \end{array}$$

The embedding $\mathbf{Sp} \hookrightarrow \mathbf{CatSp}$ also admits both adjoints and the finite product-preserving left adjoint $|-| : \mathbf{CatSp} \to \mathbf{Sp}$ is called *geometric realization*. To implement our philosophy, a categorical spectrum $\mathcal{C}$ is understood as a higher categorical structure of 'bordisms', its geometric realization $|\mathcal{C}|$ as the 'bordism spectrum' and the homotopy categories of its levels as the ordinary 'bordism categories'.

The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ can be presented by a model category. We introduce some model categories presenting the 'theory of (∞,∞)-categories' and establish their equivalence, primarily based on the existing literature. Furthermore, we derive the

Theorem (Mapping Space Formula)**.** *For any (∞,∞)-category $\mathcal{C}$, there is a natural equivalence*

$$|\mathcal{C}| \simeq \left| [n] \mapsto \mathrm{map}_{\mathbf{Cat}_{(\infty,\infty)}}(N_\omega(\mathcal{O}[n]), \mathcal{C}) \right|,$$

where the $N_\omega(\mathcal{O}[n])$ are higher categorical avatars of Street's orientals.

Finally, we define a categorical spectrum $\mathbf{Bord}_\infty$ of bordisms of closed, smooth manifolds. The 'bordism spectrum' $|\mathbf{Bord}_\infty|$ is given as $\left(\left|\mathbf{Bord}_\infty^l\right|\right)_{l\geq 0}$, where $\mathbf{Bord}_\infty^l$ is an (∞,∞)-category whose k-morphisms are $(k-l)$-dimensional bordisms. Then, $\mathrm{map}_{\mathbf{Cat}_{(\infty,\infty)}}\left(N_\omega\left(\mathcal{O}[n]\right), \mathbf{Bord}_\infty^l\right)$ is a space of $(n-l)$-dimensional manifold $[n]$-ads and the mapping space formula relates this description of $|\mathbf{Bord}_\infty|$ to Quinn's (as well as Laures and McClure's) construction of the Thom spectrum MO.

1.3 Outline

In Section 2, we review the theory of strict higher categories. This serves as a motivation and analogy for our later approach to constructing weak higher categories, as well as their structure. In Subsection 2.4, we introduce Street's *orientals* $\mathcal{O}[n]$, which will play a crucial role in Section 4.

In Section 3, we introduce the ∞-categories of weak higher categories and categorical spectra. Lurie's theory of complete Segal space objects is reviewed in Subsection 3.1 and we discuss cartesian closedness and the delooping hypothesis, including proofs that are to our knowledge novel. The ∞-category **CatSp** of categorical spectra is ultimately introduced in Subsection 3.4.

In Section 4, we discuss Barwick's n-fold complete Segal spaces, Rezk's Segal Θ_n-spaces and Verity's complicial sets as model categories that present $\mathbf{Cat}_{(\infty,n)}$. In Subsection 4.4, we introduce a nerve construction for strict n-categories, and in Subsection 4.5, we prove the mapping space formula.

The symmetrical monoidal (∞,∞)-category/connective categorical spectrum $\mathbf{Bord}_\infty$ is defined in Subsection 5.1 and future research into 'bordism-type theories' is discussed in Subsection 5.2.

In Appendix A, we discuss the theory of cartesian closed ∞-categories. This background is used in Subsection 3.1. In Appendix B, we provide a technical construction of adjunctions between sequential limits. This is used to obtain a number of adjunctions in Subsections 3.3 and 3.4.

1.4 Disclaimer on Size Issues

It should be implicitly understood that we have fixed inaccessible cardinals $\kappa_0 < \kappa_1 < \kappa_2$. The term 'small set' refers to a κ_0-small set, the term 'large set' to a κ_1-small set and the term 'very large set' to a κ_2-small set. The default size is 'large'. Expressions like $\mathbf{Cat}_\infty$ denote the large ∞-category of small ∞-categories, whereas $\mathbf{CAT}_\infty$ denotes the very large ∞-category of large large ∞-categories, if necessary. The truth is invariant under change of these inaccessible cardinals. These conventions are consistent with [Lur09b, Section 1.2.15].

1.5 Acknowledgments

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2 Strict Higher Categories

In this Section, we review the theory of strict higher categories. The discussion in Subsection 2.1 motivates our approach and the discussion in Subsections 2.2-2.3 the structure theory in Section 3 by virtue of analogy. In Subsection 2.4, we introduce Street's orientals. The reader who is already familiar may skip this Section until it is called back upon.

2.1 Enrichment vs. Internalization

There are two equivalent ways to define a small category $\mathcal{C}$. One specifies a set $\mathrm{Ob}(\mathcal{C})$ of objects, a Hom-set $\mathcal{C}(x,y)$ for any $x,y \in \mathrm{Ob}(\mathcal{C})$ and a composition map $\circ\colon \mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$ for any $x,y,z \in \mathrm{Ob}(\mathcal{C})$. The other specifies a set $\mathcal{C}_0$ of objects, a set $\mathcal{C}_1$ of morphisms, source and target maps $s,t\colon \mathcal{C}_1 \to \mathcal{C}_0$, an identity-assigning map $e\colon \mathcal{C}_0 \to \mathcal{C}_1$ and a composition map $c\colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$. Then, respective unitality and associativity axioms are imposed. These definitions admit divergent generalizations.

In the former case, we work over a monoidal category $(\mathcal{V}, \otimes, 1)$, retain a (small) set $\mathrm{Ob}(\mathcal{C})$ of objects and instead require a Hom-object $\mathcal{C}(x,y) \in \mathcal{V}$ for any $x,y \in \mathrm{Ob}(C)$ and a composition morphism $\mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \to \mathcal{C}(x,z)$ in $\mathcal{V}$ for any $x,y,z \in \mathcal{C}$, again subject to unitality and associativity axioms. This yields an ordinary category $\mathcal{V}\text{-}\mathbf{Cat}$ of (small) $\mathcal{V}$-*enriched* categories. In the latter case, we work in a finitely complete category $\mathcal{V}$ and restate the definitions verbatim with objects and morphisms in $\mathcal{V}$, yielding an ordinary category $\mathbf{Cat}(\mathcal{V})$ of categories *internal* to $\mathcal{V}$.

For a finitely complete category $\mathcal{V}$, we also regard it as a cartesian monoidal category and obtain the theories of $\mathcal{V}$-enriched categories and categories internal to $\mathcal{V}$. These both coincide with the

theory of ordinary categories if $\mathcal{V} = \mathbf{Set}$, yet are disparate in general. There is an 'internalization' functor, turning a $\mathcal{V}$-enriched category into a $\mathcal{V}$-internal category, where $\mathcal{C}_0 = \coprod_{x \in \mathrm{Ob}(\mathcal{C})} 1$ and $\mathcal{C}_1 = \coprod_{(x,y) \in \mathrm{Ob}(C) \times \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x,y)$. And, conversely, there is an 'enrichment' functor, turning a category internal to $\mathcal{V}$ into a $\mathcal{V}$-enriched category, where $\mathrm{Ob}(\mathcal{C}) = \mathcal{V}(1, \mathcal{C}_0)$ and $\mathcal{C}(x,y) = \mathcal{C}_1 \times_{\mathcal{C}_0 \times \mathcal{C}_0} 1$ for $x, y \colon 1 \to \mathcal{C}_0$. These, under mild assumptions on $\mathcal{V}$, constitute an adjunction $\mathrm{Int} \colon \mathcal{V}\text{-}\mathbf{Cat} \rightleftarrows \mathbf{Cat}(\mathcal{V}) \colon \mathrm{En}$.

This suggests that the categories internal to $\mathcal{V}$ most closely related to the $\mathcal{V}$-enriched categories are those such that $\mathcal{C}_0$ is *discrete*, i.e. a coproduct of terminal objects. Letting $\mathbf{Cat}_d(\mathcal{V})$ denote the corresponding full subcategory of $\mathbf{Cat}(\mathcal{V})$, this can be made precise:

Theorem 2.1.1 ([CFP17, Theorem 4.4])**.** *Let $\mathcal{V}$ be a finitely complete, cartesian closed and extensive category. Then,* $\mathrm{Int} \colon \mathbf{Cat}_d(\mathcal{V}) \rightleftarrows \mathcal{V}\text{-}\mathbf{Cat} \colon \mathrm{En}$ *is an adjoint equivalence.* □

2.2 The category of n-categories

If we conceptualize a category to consist of objects and morphisms between them that may be composed, then a (strict) *higher category* is conceptualized by additionally including 'morphisms between morphisms', 'morphisms between morphisms between morphisms' and so on. The language of enriched categories turns this into a definition.

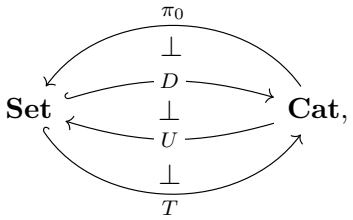
Definition 2.2.1. The (ordinary) *category* $\mathbf{Cat}_n^{\mathrm{st}}$ *of (small) strict n-categories* is defined inductively by setting $\mathbf{Cat}_0^{\mathrm{st}} \coloneqq \mathbf{Set}$, the category of (small) sets, and $\mathbf{Cat}_{n+1}^{\mathrm{st}} \coloneqq \mathbf{Cat}_n^{\mathrm{st}}\text{-}\mathbf{Cat}$ for $n \geq 0$. In particular, a *strict n-category* is a category enriched in $\mathbf{Cat}_{n-1}^{\mathrm{st}}$, for $n \geq 1$.

Remark 2.2.2. For the remainder of this section, we omit the adjective 'strict' for the sake of brevity. In future sections, it will always be made explicit.

There is an alternative 'unwound' definition, which says that an n-category consists of sets of k-morphisms $\mathcal{C}_k$ for all $0 \leq k \leq n$, source and target maps $s_{rk}, t_{rk} \colon \mathcal{C}_k \to \mathcal{C}_r$ for all $0 \leq r < k \leq n$, identity maps $1 \colon \mathcal{C}_k \to \mathcal{C}_{k+1}$ for all $0 \leq k < n$ and composition maps $\circ_r \colon \mathcal{C}_k \times_{\mathcal{C}_r} \mathcal{C}_k \to \mathcal{C}_k$ for all $0 \leq r < k \leq n$, satisfying unitality, associativity and interchange axioms. This viewpoint will be used interchangeably, and the reader is referred to [Lei04, Section 1.4] for an exposition.

Example 2.2.3. The initial n-category $\emptyset$ is the n-category having no k-morphisms for any $0 \leq k \leq n$. The terminal n-category 1 is the n-category having exactly one k-morphism for any $0 \leq k \leq n$.

Remark 2.2.4. There is an adjoint quadruple



where π_0 is the connected components, D the discrete category, U the underlying set of objects and T the codiscrete category functor. These adjunctions are (strictly) cartesian monoidal, so they inductively induce adjoint quadruples $(\pi_0)_n \dashv D_n \dashv U_n \dashv T_n$ between $\mathbf{Cat}^{\mathrm{st}}_{n+1}$ and $\mathbf{Cat}^{\mathrm{st}}_n$, for any $n \geq 0$. Explicitly, $(\pi_0)_n$ identifies n-morphisms that can be connected by a zig-zag of $(n+1)$-morphism and U_n discards all the $(n+1)$-morphisms of an $(n+1)$-category, whereas D_n equips an n-category with only the identity $(n+1)$-morphisms. The composite adjoint quadruple between $\mathbf{Cat}^{\mathrm{st}}_{n+k}$ and $\mathbf{Cat}^{\mathrm{st}}_n$ is denoted as $(\pi_0)^k_n \dashv D^k_n \dashv U^k_n \dashv T^k_n$ for any $n, k \geq 0$.

Definition 2.2.5. For an n-category $\mathcal{C}$, its *suspension* is the $(n+1)$-category $\Sigma\mathcal{C}$ with two objects $0, 1$, mapping n-categories $\Sigma\mathcal{C}(0,0) = \Sigma\mathcal{C}(1,1) = 1$, $\Sigma\mathcal{C}(1,0) = \emptyset$ and $\Sigma\mathcal{C}(0,1) = \mathcal{C}$ and the only possible composition law. This extends to an adjunction $\Sigma\colon \mathbf{Cat}^{\mathrm{st}}_n \rightleftarrows \left(\mathbf{Cat}^{\mathrm{st}}_{n+1}\right)_{*,*} : \mathrm{Hom}$.

Example 2.2.6. The 'free n-morphism' E_n is the n-category inductively defined by setting $E_0 = 1$ and $E_{n+1} = \Sigma E_n$. For $0 \leq k \leq n$, E_k regarded as an n-category represents the functor $(-)_k\colon \mathbf{Cat}^{\mathrm{st}}_n \to \mathbf{Set}$ taking an n-category to its set of k-morphisms.

Theorem 2.2.7. *For any $n \geq 0$, the category $\mathbf{Cat}^{\mathrm{st}}_n$ is (locally) finitely presentable, cartesian closed and extensive.*

Proof. The proof proceeds by induction on n, where the case $n = 0$ is standard. The property of being finitely complete, cartesian closed and extensive is stable under enrichment by [CFP17, Corollary 2.7].

Though cocompleteness is also stable under enrichment, we give a more concrete argument. Namely, $\mathbf{Cat}^{\mathrm{st}}_n$ can be inductively exhibited as a reflective localization of $\mathbf{s}^n\mathbf{Set} \coloneqq \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{Set}\right)$, the category of n-fold simplicial sets. For $n = 1$, this is the adjunction $h\colon \mathbf{sSet} \rightleftarrows \mathbf{Cat} : N$. In the inductive step, abstract nonsense[1] implies that $\mathbf{Cat}^{\mathrm{st}}_n = \mathbf{Cat}^{\mathrm{st}}_{n-1}\text{-}\mathbf{Cat}$ is a reflective localization of $\mathbf{s}^{n-1}\mathbf{Set}\text{-}\mathbf{Cat}$. This again is a reflective localization of $\mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{Cat}\right)$. Indeed, for any small category $\mathcal{C}$, the category $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})\text{-}\mathbf{Cat}$ is equivalent to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{Cat})$ on those functors whose copresheaf of objects is constant and a reflection is given by

$$(F\colon \mathcal{C} \to \mathbf{Cat}) \mapsto \left(F'\colon \mathcal{C} \to \mathbf{Cat},\, c \mapsto Fc \sqcup_{D(\mathrm{Ob}(Fc))} D\left(\mathrm{colim}_{\mathcal{C}}\, \mathrm{Ob}(F)\right)\right), \tag{1}$$

where this pushout exists by the case $n = 1$. Then, applying the case $n = 1$ once more, we see that $\mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{Cat}\right)$ is a reflective localization of $\mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{sSet}\right) = \mathbf{s}^n\mathbf{Set}$. In total, $\mathbf{Cat}^{\mathrm{st}}_n$ is a reflective localization of $\mathbf{s}^n\mathbf{Set}$, hence cocomplete.

Finally, *filtered* colimits in $\mathbf{Cat}^{\mathrm{st}}_n$ can be constructed explicitly. Indeed, given a filtered diagram $\mathcal{C}_\bullet\colon \mathcal{I} \to \mathbf{Cat}^{\mathrm{st}}_n$, its colimit $\mathcal{C}$ has $\mathrm{Ob}(\mathcal{C}) = \mathrm{colim}_{\mathcal{I}}\, \mathrm{Ob}(\mathcal{C}_\bullet)$ and

$$\mathcal{C}([x,i],[y,j]) = \operatorname*{colim}_{(\tau\colon k \to k') \in \mathcal{I}_{k/}} \mathcal{C}_{k'}((\tau\varphi)_*(x), (\tau\psi)_*(y))$$

[1] The construction $\mathcal{V} \mapsto \mathcal{V}\text{-}\mathbf{Cat}$ is 2-functorial.

(this is a filtered colimit in $\mathbf{Cat}_{n-1}^{\mathrm{st}}$) for any two $[x,i],[y,j] \in \mathrm{Ob}(\mathcal{C})$, where we have chosen a $k \in \mathcal{I}$ and morphisms $\varphi\colon i \to k$ and $\psi\colon j \to k$ in $\mathcal{I}$, which is possible since $\mathcal{I}$ is filtered (and the colimit is independent of their choice up to canonical isomorphism). From this description, we infer that finite n-categories are compact objects in $\mathbf{Cat}_n^{\mathrm{st}}$ and that every n-category is the colimit over its directed poset of finite subcategories. Thus, $\mathbf{Cat}_n^{\mathrm{st}}$ is (locally) finitely presentable. □

Corollary 2.2.8. *The category $\mathbf{Cat}_n^{\mathrm{st}}$ has a canonical enrichment as a (large) $(n+1)$-category.* □

Remark 2.2.9. The interpretation of $\mathbf{Cat}_n^{\mathrm{st}}$ as a subcategory of $\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{Set}\right)$ can be seen as an instance of the comparison between internalization and enrichment discussed in Subsection 2.1. This point is expanded upon in Remark 3.2.6 and Subsection 4.4.

Definition 2.2.10. A *strict (n,r)-category*, $0 \le r \le n$, is a strict n-category whose k-morphisms for $r < k \le n$ are *invertible*, i.e. admit strict inverses w.r.t. $\circ_{k-1}$. The category $\mathbf{Cat}_{(n,r)}^{\mathrm{st}}$ is the full subcategory of $\mathbf{Cat}_n^{\mathrm{st}}$ on the strict (n,r)-categories.

Proposition 2.2.11. *For $1 \le r \le n$, we have $\mathbf{Cat}_{(n,r)}^{\mathrm{st}} = \mathbf{Cat}_{(n-1,r-1)}^{\mathrm{st}}\text{-}\mathbf{Cat}$.* □

Remark 2.2.12. For $0 \le r \le n$ and $k \ge 0$, the adjoint quadruple $(\pi_0)_n^k \dashv D_n^k \dashv U_n^k \dashv T_n^k$ restricts to an adjoint quadruple $(\pi_0)_{(n,r)}^k \dashv D_{(n,r)}^k \dashv U_{(n,r)}^k \dashv T_{(n,r)}^k$ between $\mathbf{Cat}_{(n+k,r)}^{\mathrm{st}}$ and $\mathbf{Cat}_{(n,r)}^{\mathrm{st}}$.

Proposition 2.2.13. *For $0 \le r < n$, the inclusion $J_{(n,r)}\colon \mathbf{Cat}_{(n,r)}^{\mathrm{st}} \hookrightarrow \mathbf{Cat}_{(n,r+1)}^{\mathrm{st}}$ is part of an adjoint triple $L_{(n,r)} \dashv J_{(n,r)} \dashv C_{(n,r)}$.*

Proof. This is proven by induction on n, the case $n = 0$ being empty. If $r > 0$, the adjoints of $J_{(n-1,r-1)}\colon \mathbf{Cat}_{(n-1,r-1)}^{\mathrm{st}} \hookrightarrow \mathbf{Cat}_{(n-1,r)}^{\mathrm{st}}$ induce the adjoints of $J_{(n,r)}\colon \mathbf{Cat}_{(n,r)}^{\mathrm{st}} \hookrightarrow \mathbf{Cat}_{(n,r+1)}^{\mathrm{st}}$ via enrichment, so we may assume $r = 0$. The right adjoint $C_{(n,0)}\colon \mathbf{Cat}_{(n,1)}^{\mathrm{st}} \to \mathbf{Cat}_{(n,0)}^{\mathrm{st}}$ takes an $(n,1)$-category $\mathcal{C}$ to the $(n,0)$-category with the same set of objects and each Hom-$(n-1,0)$-category replaced with its full subcategory on the invertible 1-morphisms.

To define the left adjoint, let J denote the 1-category consisting of two objects and a unique isomorphism between them, and let $E_1 \to J$ classify that isomorphism. The left adjoint $L_{(n,0)}\colon \mathbf{Cat}_{(n,1)}^{\mathrm{st}} \to \mathbf{Cat}_{(n,0)}^{\mathrm{st}}$ takes an $(n,1)$-category $\mathcal{C}$ to the $(n,0)$-category $L_{(n,0)}(\mathcal{C}) = \mathcal{C} \sqcup_{\coprod_{E_1 \to \mathcal{C}} E_1} \coprod_{E_1 \to \mathcal{C}} J_1$. This pushout can be computed in $\mathbf{Cat}_{(n,1)}^{\mathrm{st}}$ (since $\mathbf{Cat}_{(n,1)}^{\mathrm{st}} \hookrightarrow \mathbf{Cat}_n^{\mathrm{st}}$ is a left adjoint and thus preserves colimits), so that it only remains to consider the 1-morphisms to establish that $L_{(n,0)}\mathcal{C}$ is indeed an $(n,0)$-category. Thus, we may apply the colimit-preserving functor $U_1^{n-1}\colon \mathbf{Cat}_n^{\mathrm{st}} \to \mathbf{Cat}_1^{\mathrm{st}}$ and use an explicit model for colimits of 1-categories to see that the 1-morphisms in $L_{(n,0)}(\mathcal{C})$ are generated under composition by those in $\mathcal{C}$ and those in $\coprod_{E_1 \to \mathcal{C}} J_1$, either of which are invertible in $L_{(n,0)}\mathcal{C}$ by construction. The universal property is then easy to verify since a 1-morphism $E_1 \to \mathcal{C}$ in an n-category $\mathcal{C}$ is invertible if and only if it extends along $E_1 \to J$. □

Remark 2.2.14. Thus, $L_{(n,r)}$ formally adjoins inverses to $(r+1)$-morphisms and $C_{(n,r)}$ discards non-invertible $(r+1)$-morphisms of an $(n, r+1)$-category. The composite adjoint triple between $\mathbf{Cat}^{\mathrm{st}}_{(n,r+k)}$ and $\mathbf{Cat}^{\mathrm{st}}_{(n,r)}$ is denoted $L^k_{(n,r)} \dashv J^k_{(n,r)} \dashv C^k_{(n,r)}$ for any $n, r \geq 0$ and $0 \leq k \leq n-r$.

2.3 The category of ω-categories

The 'unwound' definition of an n-category given after Definition 2.2.1 generalizes to an 'unwound' definition of an ω-category, simply by removing the upper bound n in all places. This translates into the following algebraic definition (see [Lei04, Proposition 1.4.12]).

Definition 2.3.1. The *category* $\mathbf{Cat}^{\mathrm{st}}_{\omega}$ *of (small) strict* ω*-categories* is the limit (in $\mathbf{CAT}$) of

$$\ldots \xrightarrow{U_3} \mathbf{Cat}^{\mathrm{st}}_3 \xrightarrow{U_2} \mathbf{Cat}^{\mathrm{st}}_2 \xrightarrow{U_1} \mathbf{Cat} \xrightarrow{U} \mathbf{Set}.$$

In particular, a *strict* ω*-category* is a sequence $(\mathcal{C}_n)_{n\geq 0}$, where $\mathcal{C}_n$ is an n-category and $U_n(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$.

Remark 2.3.2. The forgetful functors U_n, $n \geq 0$ are isofibrations, so this strict limit is, up to equivalence, also a pseudolimit/a homotopy limit with respect to the canonical model structure on $\mathbf{CAT}$.

The process of enrichment can not be iterated any further:

Proposition 2.3.3. *There is a canonical isomorphism* $\mathbf{Cat}^{\mathrm{st}}_{\omega}\text{-}\mathbf{Cat} \cong \mathbf{Cat}^{\mathrm{st}}_{\omega}$.

Proof. Indeed,

$$\mathbf{Cat}^{\mathrm{st}}_{\omega}\text{-}\mathbf{Cat} \cong \left(\lim_{n\geq 0} \mathbf{Cat}^{\mathrm{st}}_n\right)\text{-}\mathbf{Cat} \cong \lim_{n\geq 0}\left(\mathbf{Cat}^{\mathrm{st}}_n\text{-}\mathbf{Cat}\right) \cong \lim_{n\geq 0} \mathbf{Cat}^{\mathrm{st}}_{n+1} \cong \mathbf{Cat}^{\mathrm{st}}_{\omega}.$$

The second isomorphism is a consequence of the fact that the U_n, $n \geq 0$ preserve products. □

Proposition 2.3.4. *For any* $n \geq 0$*, the fully faithful inclusions* $D^k_n, T^k_n \colon \mathbf{Cat}^{\mathrm{st}}_n \hookrightarrow \mathbf{Cat}^{\mathrm{st}}_{n+k}$, $k \geq 0$ *induce fully faithful inclusions* $D^{\omega}_n, T^{\omega}_n \colon \mathbf{Cat}^{\mathrm{st}}_n \hookrightarrow \mathbf{Cat}^{\mathrm{st}}_{\omega}$ *that are part of an adjoint quadruple* $(\pi_0)^{\omega}_n \dashv D^{\omega}_n \dashv U^{\omega}_n \dashv T^{\omega}_n$*, where* $U^{\omega}_n \colon \mathbf{Cat}^{\mathrm{st}}_{\omega} \to \mathbf{Cat}^{\mathrm{st}}_n$ *denotes the canonical projection and* $(\pi_0)^{\omega}_n \coloneqq (\pi_0)_n U^{\omega}_{n+1}$.

Proof. For $\mathcal{C} \in \mathbf{Cat}^{\mathrm{st}}_n$ and $\mathcal{D} \in \mathbf{Cat}^{\mathrm{st}}_{\omega}$, we calculate

$$\mathbf{Cat}^{\mathrm{st}}_{\omega}\left(D^{\omega}_n\mathcal{C}, \mathcal{D}\right) \cong \lim_{k\geq 0} \mathbf{Cat}^{\mathrm{st}}_{n+k}\left(D^k_n\mathcal{C}, \mathcal{D}_{n+k}\right) \cong \lim_{k\geq 0} \mathbf{Cat}^{\mathrm{st}}_n\left(\mathcal{C}, U^k_n\mathcal{D}_{n+k}\right) \cong \mathbf{Cat}^{\mathrm{st}}_n\left(\mathcal{C}, \mathcal{D}_n\right).$$

The other adjunctions are verified similarly, one also using that $(\pi_0)_n(\pi_0)_{n+1} = (\pi_0)_n U_{n+1}$ for $n \geq 0$, which is verified for $n = 0$ by hand and then follows by induction. □

Lemma 2.3.5. *For any $\mathcal{C} = (\mathcal{C}_n)_{n\geq 0} \in \mathbf{Cat}_\omega^{\mathrm{st}}$, the counits of the adjunctions $D_n^\omega \dashv U_n^\omega$ induce a natural isomorphism* $\operatorname{colim}_{n\geq 0} D_n^\omega U_n^\omega \mathcal{C} = \operatorname{colim}_{n\geq 0} D_n^\omega \mathcal{C}_n \xrightarrow{\sim} \mathcal{C}$.

Proof. For any ω-category $\mathcal{D} = (\mathcal{D}_n)_{n\geq 0}$, we calculate

$$\mathbf{Cat}_\omega^{\mathrm{st}}\left(\operatorname*{colim}_{n\geq 0} D_n^\omega \mathcal{C}_n, \mathcal{D}\right) \cong \lim_{n\geq 0} \mathbf{Cat}_\omega^{\mathrm{st}}\left(D_n^\omega \mathcal{C}_n, \mathcal{D}\right) \cong \lim_{n\geq 0} \mathbf{Cat}_n^{\mathrm{st}}\left(\mathcal{C}_n, \mathcal{D}_n\right) \cong \mathbf{Cat}_\omega^{\mathrm{st}}\left(\mathcal{C}, \mathcal{D}\right). \qquad \square$$

Theorem 2.3.6. *The category $\mathbf{Cat}_\omega^{\mathrm{st}}$ is (locally) finitely presentable and cartesian closed. In particular, it has a canonical enrichment as a (large) ω-category.*

Proof. The category $\mathbf{Cat}_\omega^{\mathrm{st}}$ is (co-)complete and its (co-)limits are computed levelwise, by abstract nonsense. Fix internal Hom-functors $\mathrm{Fun}_n \colon \left(\mathbf{Cat}_n^{\mathrm{st}}\right)^{op} \times \mathbf{Cat}_n^{\mathrm{st}} \to \mathbf{Cat}_n^{\mathrm{st}}$ for all $n \geq 0$, and ω-categories $\mathcal{C} = (\mathcal{C}_n)_{n\geq 0}$, $\mathcal{D} = (\mathcal{D}_n)_{n\geq 0}$ and $\mathcal{E} = (\mathcal{E}_n)_{n\geq 0}$. For any $n \geq 0$, there is a composite

$$U_n \operatorname{Fun}_{n+1}(\mathcal{D}_{n+1}, \mathcal{E}_{n+1}) \times U_n \mathcal{D}_{n+1} \cong U_n \left(\operatorname{Fun}_{n+1}(\mathcal{D}_{n+1}, \mathcal{E}_{n+1}) \times \mathcal{D}_{n+1}\right) \to U_n \mathcal{E}_{n+1},$$

adjoint to a morphism $U_n \operatorname{Fun}_{n+1}(\mathcal{D}_{n+1}, \mathcal{E}_{n+1}) \to \operatorname{Fun}_n(\mathcal{D}_n, \mathcal{E}_n)$. This yields a sequence

$$\ldots \longrightarrow U_n^2 \operatorname{Fun}_{n+2}(\mathcal{D}_{n+2}, \mathcal{E}_{n+2}) \longrightarrow U_n \operatorname{Fun}_{n+1}(\mathcal{D}_{n+1}, \mathcal{E}_{n+1}) \longrightarrow \operatorname{Fun}_n(\mathcal{D}_n, \mathcal{E}_n).$$

Inductively (use Remark 2.3.2), we can choose limits of these sequences such that they define an object $\operatorname{Fun}_\omega(\mathcal{D}, \mathcal{E}) \coloneqq \left(\lim_{m\geq n} U_n^{m-n} \operatorname{Fun}_m(\mathcal{D}_m, \mathcal{E}_m)\right)_{n\geq 0}$. To check the universal property, we calculate

$$\begin{aligned}
\mathbf{Cat}_\omega^{\mathrm{st}}\left(\mathcal{C}, \operatorname{Fun}_\omega(\mathcal{D}, \mathcal{E})\right) &\cong \lim_{n\geq 0} \mathbf{Cat}_n^{\mathrm{st}}\left(\mathcal{C}_n, \lim_{m\geq n} U_n^{m-n} \operatorname{Fun}_m(\mathcal{D}_m, \mathcal{E}_m)\right) \\
&\cong \lim_{n\geq 0} \lim_{m\geq n} \mathbf{Cat}_n^{\mathrm{st}}\left(\mathcal{C}_n, U_n^{m-n} \operatorname{Fun}_m(\mathcal{D}_m, \mathcal{E}_m)\right) \\
&\cong \lim_{n\geq 0} \lim_{m\geq n} \mathbf{Cat}_m^{\mathrm{st}}\left(D_n^{m-n}\mathcal{C}_n \times \mathcal{D}_m, \mathcal{E}_m\right) \\
&\cong \lim_{m\geq 0} \lim_{n\leq m} \mathbf{Cat}_m^{\mathrm{st}}\left(D_n^{m-n}\mathcal{C}_n \times \mathcal{D}_m, \mathcal{E}_m\right) \\
&\cong \lim_{m\geq 0} \mathbf{Cat}_m^{\mathrm{st}}\left(\mathcal{C}_m \times \mathcal{D}_m, \mathcal{E}_m\right) \\
&\cong \mathbf{Cat}_\omega^{\mathrm{st}}\left(\mathcal{C} \times \mathcal{D}, \mathcal{E}\right).
\end{aligned}$$

Thus, $\mathbf{Cat}_\omega^{\mathrm{st}}$ is cartesian closed.

Lastly, $D_n^\omega \colon \mathbf{Cat}_n^{\mathrm{st}} \to \mathbf{Cat}_\omega^{\mathrm{st}}$ is left adjoint to U_n^ω and this functor preserves (filtered) colimits, hence D_n^ω preserves compact objects. Thus, combining Theorem 2.2.7 and Lemma 2.3.5, $\mathbf{Cat}_\omega^{\mathrm{st}}$ is generated under filtered colimits by compact objects, hence (locally) finitely presentable. $\square$

Remark 2.3.7. It can also be shown that $\mathbf{Cat}_\omega^{\mathrm{st}}$ is generated under arbitrary colimits by the E_n, $n \geq 0$.

Proposition 2.3.8. *The limit of the induced diagram*

$$\dots \xrightarrow{U_{(r,r+2)}} \mathbf{Cat}^{\mathrm{st}}_{(r+2,r)} \xrightarrow{U_{(r+1,r)}} \mathbf{Cat}^{\mathrm{st}}_{(r+1,r)} \xrightarrow{U_{(r,r)}} \mathbf{Cat}^{\mathrm{st}}_{r}.$$

is the full subcategory $\mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}$ *of* $\mathbf{Cat}^{\mathrm{st}}_{\omega}$ *on those* ω*-categories in which all* k*-morphisms for* $k > r$ *are invertible (w.r.t.* $\circ_{k-1}$*).* □

Remark 2.3.9. For $0 \leq r \leq n$, the adjoint quadruple $(\pi_0)^{\omega}_{n} \dashv D^{\omega}_{n} \dashv U^{\omega}_{n} \dashv T^{\omega}_{n}$ restricts to an adjoint quadruple $(\pi_0)^{\omega}_{(n,r)} \dashv D^{\omega}_{(n,r)} \dashv U^{\omega}_{(n,r)} \dashv T^{\omega}_{(n,r)}$ between $\mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}$ and $\mathbf{Cat}^{\mathrm{st}}_{(n,r)}$.

Proposition 2.3.10. *For* $r \geq 1$*, we have* $\mathbf{Cat}^{\mathrm{st}}_{(\omega,r)} = \mathbf{Cat}^{\mathrm{st}}_{(\omega,r-1)}\text{-}\mathbf{Cat}$. □

Proposition 2.3.11. *There is a canonical identification* $\mathbf{Cat}^{\mathrm{st}}_{\omega} \cong \lim_{r\geq 0} \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}$.

Proof. Indeed, we have

$$\lim_{r\geq 0} \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)} \cong \lim_{r\geq 0} \lim_{n\geq r} \mathbf{Cat}^{\mathrm{st}}_{(n,r)} \cong \lim_{n\geq 0} \lim_{k\leq n} \mathbf{Cat}^{\mathrm{st}}_{(n,r)} \cong \lim_{n\geq 0} \mathbf{Cat}^{\mathrm{st}}_{(n,n)} \cong \mathbf{Cat}^{\mathrm{st}}_{\omega}.$$ □

Proposition 2.3.12. *For any* $r \geq 0$*, the inclusion* $J_{(\omega,r)} \colon \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)} \hookrightarrow \mathbf{Cat}^{\mathrm{st}}_{(\omega,r+1)}$ *is part of an adjoint triple* $L_{(\omega,r)} \dashv J_{(\omega,r)} \dashv C_{(\omega,r)}$.

Proof. For $0 \leq r < n$, the natural isomorphism $U_{(n,r)}L_{(n+1,r)} \cong L_{(n,r)}U_{(n,r+1)}$ (these are left adjoint to $T_{(n,r)}J_{(n+1,r)} = J_{(n,r)}T_{(n,r+1)}$) can be upgraded to an equality (use Remark 2.3.2). This induces a functor $L_{(\omega,r)} \colon \mathbf{Cat}^{\mathrm{st}}_{(\omega,r+1)} \to \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}$, $(\mathcal{C}_n)_{n\geq r+1} \to \left(L_{(n,r)}\mathcal{C}_n\right)_{n\geq r+1}$ and we calculate

$$\begin{aligned}\mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}\left(L_{(\omega,r)}\mathcal{C}, \mathcal{D}\right) &\cong \lim_{n\geq r+1} \mathbf{Cat}^{\mathrm{st}}_{(n,r)}\left(L_{(n,r)}\mathcal{C}_n, \mathcal{D}_n\right) \\ &\cong \lim_{n\geq r+1} \mathbf{Cat}^{\mathrm{st}}_{(n,r+1)}\left(\mathcal{C}_n, J_{(n,r)}\mathcal{D}_n\right) \cong \mathbf{Cat}^{\mathrm{st}}_{(\omega,r+1)}\left(\mathcal{C}, J_{(\omega,r)}\mathcal{D}\right), \qquad \forall \mathcal{C} \in \mathbf{Cat}^{\mathrm{st}}_{(\omega,r+1)},\ \mathcal{D} \in \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}.\end{aligned}$$

The right adjoint $C_{(\omega,r)} \colon \mathbf{Cat}^{\mathrm{st}}_{(\omega,r+1)} \to \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)}$ is analogous, since $U_{(n,r)}C_{(n+1,r)} = C_{(n,r)}U_{(n,r+1)}$. □

Then, similar arguments also yield the following two Propositions.

Proposition 2.3.13. *For any* $r \geq 0$*, the inclusion* $J^{\omega}_{(\omega,r)} \colon \mathbf{Cat}^{\mathrm{st}}_{(\omega,r)} \hookrightarrow \mathbf{Cat}^{\mathrm{st}}_{\omega}$ *is part of an adjoint triple* $L^{\omega}_{(\omega,r)} \dashv J^{\omega}_{(\omega,r)} \dashv C^{\omega}_{(\omega,r)}$. □

Proposition 2.3.14. *For any* $\mathcal{C} \in \mathbf{Cat}^{\mathrm{st}}_{\omega}$*, the counits of the adjunctions* $J^{\omega}_{(\omega,r)} \dashv C^{\omega}_{(\omega,r)}$ *induce a natural isomorphism* $\mathrm{colim}_{r\geq 0} J^{\omega}_{(\omega,r)}C^{\omega}_{(\omega,r)}\mathcal{C} \xrightarrow{\sim} \mathcal{C}$. □

2.4 Orientals

The data governing strict higher categories is of a 'globular' shape. In homotopy theory, we prefer to work with *simplicial* shapes. To translate, we need to incarnate the n-simplex as an n-category. These categories are known as the *orientals* and were introduced in [Str87]. Their combinatorial complexity is notorious, so we review Steiner's theory of augmented directed chain complexes to give an accessible definition. This serves a primarily expository purpose as the technical results will all be siphoned from the literature. For a more complete exposition, see [Ste04] or [GOR23, Section 2].

Definition 2.4.1 ([Ste04, Definition 2.2])**.** An *augmented directed chain complex* is a triple $(C_\bullet, C_\bullet^+, \varepsilon)$, where $(C_\bullet, \varepsilon)$ is a non-negative augmented chain complex of abelian groups and $C_\bullet^+ = (C_n^+)_{n\geq 0}$ is a collection of submonoids $C_n^+ \leq \mathcal{C}_n$ for all $n \geq 0$, called the *positivity submonoids*. A map of augmented directed chain complexes is a chain map that commutes with the augmentations and preserves the positivity submonoids. This describes the category **adCh** of augmented directed chain complexes.

Remark 2.4.2. The category **adCh** is both complete and cocomplete.

Definition 2.4.3 ([Ste04, Definition 2.8])**.** Let $C = (C_\bullet, C_\bullet^+, \varepsilon)$ be an augmented directed chain complex. The ω-category νC has as n-morphisms $(\nu C)_n$ the set of sequences

$$x = \left(x_n^-, x_n^+ | x_{n-1}^-, x_{n-1}^+ | \dots | x_0^-, x_0^+\right),$$

such that $x_i^\pm \in C_i^+$ for $0 \leq i \leq n$, $\partial\left(x_i^\pm\right) = x_{i-1}^+ - x_{i-1}^-$ for $0 < i \leq n$, $\varepsilon\left(x_0^\pm\right) = 1$ and $x_n^+ = x_n^-$. The source and target maps are defined by

$$s_{kn}\left(x_n^\pm | \dots | x_0^\pm\right) = \left(x_k^-, x_k^- | \dots | x_0^\pm\right), \qquad t_{kn}\left(x_n^\pm | \dots | x_0^\pm\right) = \left(x_k^+, x_k^+ | \dots | x_0^\pm\right), \qquad 0 \leq k < n.$$

The identity of $x = \left(x_n^\pm | \dots | x_0^\pm\right) \in (\nu C)_n$ is given by $1_x = \left(0, 0 | x_n^\pm | \dots | x_0^\pm\right) \in (\nu C)_{n+1}$. For $0 \leq k < n$ and two n-morphisms x, y, the composite $y \circ_k x$ is defined if and only if $x_k^+ = y_k^-$ and $x_r^\pm = y_r^\pm$ for $0 \leq r < k$, in which case it is given by

$$y \circ_k x = (x_n^\pm + y_n^\pm | \dots | x_{k+1}^\pm + y_{k+1}^\pm | x_k^-, y_k^+ | x_{k-1}^\pm | \dots | x_0^\pm).$$

This extends to a functor $\nu\colon \mathbf{adCh} \to \mathbf{Cat}_\omega^{\mathrm{st}}$.

Remark 2.4.4. In fact, there is an adjunction $\mu\colon \mathbf{Cat}_\omega^{\mathrm{st}} \rightleftarrows \colon \mathbf{adCh}\colon \nu$ (see [Ste04, Theorem 2.11]), but we have no need for the left adjoint.

Definition 2.4.5 ([Ste04, Definition 3.1])**.** Let C be an augmented directed chain complex. A *basis* for C is a collection $(B_n)_{n\geq 0}$ of subsets $B_n \subseteq C_n$ such that C_n is a free abelian group on B_n and C_n^+ is a free abelian monoid on B_n, for all $n \geq 0$.

For an element $b \in B_n$, inductively define $\langle b \rangle_m^{\pm}$ for $m \leq n$ by setting $\langle b \rangle_n^{\pm} = b$ and letting $\langle b \rangle_m^{+}$ resp. $\langle b \rangle_m^{-}$ be the sum of the terms in the basis expansion of $\partial \langle b \rangle_{m+1}^{\pm}$ with positive resp. negative coefficients. The basis is *unital* if $\varepsilon\left(\langle b \rangle_0^{\pm}\right) = 1$ for all $b \in B_n$, $n \geq 0$, in which case $\langle b \rangle \in (\nu C)_n$ for any basis element b and this morphism is called an *atom* of νC.

Definition 2.4.6 ([Ste04, Example 3.8])**.** Let K be an ordered (abstract) simplicial complex, V its set of vertices. The augmented directed chain complex $C^{\mathrm{simp}}(K)$ has C_n the free abelian group on the n-simplices $(v_0, \ldots, v_n)$ of K, C_n^+ the free abelian monoid on these n-simplices, differential given by $\partial(v_0, \ldots, v_n) = \sum_{i=0}^{n} (-1)^i d_i(v_0, \ldots, v_n)$ and augmentation $\varepsilon \colon C_0 \to \mathbb{Z}$, $(v) \mapsto 1$. This extends to a functor $C^{\mathrm{simp}} \colon \mathbf{OSC} \to \mathbf{adCh}$ on the category of ordered simplicial complexes.

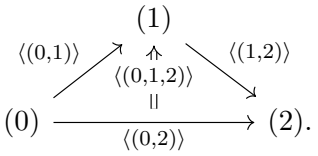
Proposition 2.4.7 ([Ste04, Example 3.8])**.** *Let K be an ordered simplicial complex. The basis of $C^{\mathrm{simp}}(K)$ just constructed is unital.* □

Definition 2.4.8 ([Ste07, Theorem 3.2])**.** The *cosimplicial oriental* $\mathcal{O} \colon \Delta \to \mathbf{Cat}_\omega^{\mathrm{st}}$ is the composite

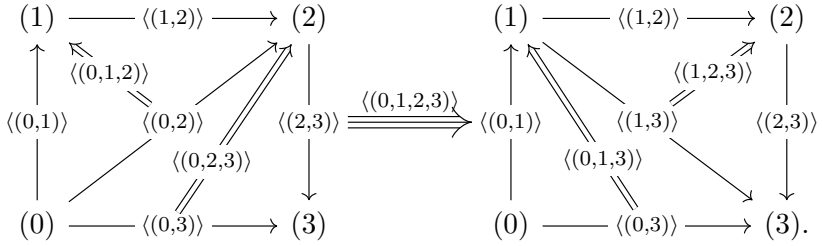
$$\Delta \longrightarrow \mathbf{OSC} \xrightarrow{C^{\mathrm{simp}}} \mathbf{adCh} \xrightarrow{\nu} \mathbf{Cat}_\omega^{\mathrm{st}}.$$

The unlabeled functor regards $[n]$ as the ordered simplicial complex Δ^n. This functor induces an adjunction $\mathcal{O} \colon \mathbf{sSet} \rightleftarrows \colon \mathbf{Cat}_\omega^{\mathrm{st}} \colon N_{\mathcal{O}}$, whose right adjoint $N_{\mathcal{O}}$ is called the *Street nerve* ([Str87, Section 5]). The *n-th oriental* is $\mathcal{O}[n] \coloneqq \mathcal{O}\left(\Delta^n\right) = \nu C^{\mathrm{simp}}\left(\Delta^n\right)$.

The first two orientals $\mathcal{O}[0] = E_0$ and $\mathcal{O}[1] = E_1$ are cells and $\mathcal{O}[2]$ is a (non-commutative) triangle



The oriental $\mathcal{O}[3]$ is already quite complicated. It can be pictured as



The left- and right-hand side should be imagined as the front- and back-side of a 3-simplex. The 0- and 1- as well as one half of the 2-dimensional atoms is depicted on either side. The unique 3-dimensional

atom is a 3-morphism whose source resp. target are the 'composite' 2-morphisms on the left- resp. right-hand side. These are not just composites of the 2-dimensional atoms, e.g. $s_{23}\langle(0,1,2,3)\rangle$ is $\big(1_{\langle(2,3)\rangle} \circ_0 \langle(0,1,2)\rangle\big) \circ_1 \langle(0,2,3)\rangle$. As n grows, it becomes increasingly harder to write the source and target of $\langle(0,\dots,n)\rangle$ as a composite of lower-dimensional atoms, see [Str87, Page 330/331].

In general, $\mathcal{O}[n]$ should be understood as the 'free n-category' on the n-simplex Δ^n. Its objects are given by the 0-simplices of Δ^n, its 1-morphisms are freely generated under composition by the edges of Δ^n, etc.. Though it is hard to give an inductive definition a priori, it is at least possible to obtain such a description *a fortiori*. Indeed, we have $\mathcal{O}(\partial\Delta^n) = U_{n-1}\mathcal{O}[n]$ ([Str87, Lemma 5.1], or [Ste04, Theorem 6.1]) and the n-dimensional atom is the unique non-identity n-morphism in $\mathcal{O}[n]$ ([Ste04, Theorem 6.2]). Thus, we can start with $\mathcal{O}[0] = E_0$ and, in the inductive step, obtain $\mathcal{O}[n]$ by gluing copies of $\mathcal{O}[n-1]$ along copies of $\mathcal{O}[n-2]$ according to the shape of $\partial\Delta^n$ and then glue in an n-morphism from the 'composite of the odd faces' to the 'composite of the even faces' to obtain $\mathcal{O}[n]$. In particular, a functor $\mathcal{O}[n] \to \mathcal{C}$ into an ω-category $\mathcal{C}$ can be given by inductively specifying images of the atoms compatible with the source and target maps.

Furthermore, there is a canonical identification $\pi_1^{n-1}\mathcal{O}[n] = [n]$. Thus, for any 1-category $\mathcal{C}$, morphisms $\mathcal{O}[n] \to \mathcal{C}$ are the same as morphisms $[n] \to \mathcal{C}$, hence $N_{\mathcal{O}}(\mathcal{C}) = N(\mathcal{C})$ and the Street nerve extends the ordinary nerve. In contrast to the ordinary nerve, however, the Street nerve is *not* fully faithful on n-categories for any $n \geq 2$, but compare Theorem 4.3.4.

3 Higher ∞-Categories

In this Section, we introduce higher ∞-categories. In Subsection 3.1, we present a general machinery for constructing (a variant of) 'enriched ∞-categories', then we apply it in Subsection 3.2 to construct the theory of (∞, n)-categories. Then, the theory of (∞, ∞)-categories in constructed in Subsection 3.3 and the theory of categorical spectra in Subsection 3.4. From this point onward, we freely use the language of ∞-categories (meaning Joyal's quasicategories), as introduced in [Lan21] or [Lur09b].

3.1 Complete Segal Space Objects

The discussion in Subsection 2.1 suggests that we can take an *internal* approach to developing 'enriched ∞-categories'. This is achieved by Lurie's theory of complete Segal space objects, though the passage from sets to spaces results in replacing the discreteness condition with a more involved *completeness* condition. The majority of this Subsection is based on [Lur09a, Sections 1.1-1.2]. Towards the end, we discuss cartesian closedness and the delooping hypothesis.

First, we introduce the general analogue of an internal category in an ∞-category $\mathcal{C}$.

Definition 3.1.1. A *simplicial object* in an ∞-category $\mathcal{C}$ is a functor $X_\bullet \colon N(\Delta)^{op} \to \mathcal{C}$. The ∞-category of simplicial objects in $\mathcal{C}$ is the ∞-category $\mathrm{Fun}(N(\Delta)^{op}, \mathcal{C})$.

Notation 3.1.2 ([Lur09a, Notation 1.1.7])**.** Let $X_\bullet$ be a simplicial object in an ∞-category $\mathcal{C}$, K a simplicial set and $\Delta_{/K}$ the (strict) category of simplices of K. The limit of $N(\Delta_{/K})^{op} \to N(\Delta)^{op} \to \mathcal{C}$, if it exists, is denoted $X(K)$. Intuitively, $X(K)$ is the object of K-indexed diagrams in $X_\bullet$.

Remark 3.1.3. If the simplicial set K is *regular*, i.e. faces of non-degenerate simplices are non-degenerate (e.g. if K is obtained from an ordered simplicial complex), then the inclusion $\Delta^{nd}_{/K} \hookrightarrow \Delta_{/K}$ of the full subcategory on non-degenerate simplices is a right adjoint (the left adjoint takes a simplex to its unique factorization through a non-degenerate simplex, though its functoriality is subtle), so $N\left(\Delta^{nd}_{/K}\right)^{op} \hookrightarrow N\left(\Delta_{/K}\right)^{op}$ is coinitial[2] and $X(K)$ can be computed as a limit over $N\left(\Delta^{nd}_{/K}\right)^{op}$ instead.

Remark 3.1.4. If $\mathcal{C}$ is complete, this construction is contravariantly functorial in K, more precisely $X_\bullet$ right Kan extends to a functor $X(-) \colon N(\mathbf{sSet})^{op} \to \mathcal{C}$ mapping $K \mapsto X(K)$.

Definition 3.1.5. A *category object* in a finitely complete ∞-category $\mathcal{C}$ is a simplicial object

$$X_\bullet \colon N(\Delta)^{op} \to \mathcal{C}$$

such that the spine inclusion $\mathrm{Sp}^n \hookrightarrow \Delta^n$ induces an equivalence $X(\Delta^n) \xrightarrow{\sim} X(\mathrm{Sp}^n)$ for any $n \geq 2$. The ∞-category $\mathbf{Cat}(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}(N(\Delta)^{op}, \mathcal{C})$ on the category objects of $\mathcal{C}$.

Remark 3.1.6. This condition is known as the *Segal condition*. The map $X(\Delta^n) \to X(\mathrm{Sp}^n)$ can be identified with the canonical map $X_n \to X_1 \times_{X_0} \ldots \times_{X_0} X_1$ induced by the face maps.

Proposition 3.1.7. *If $\mathcal{C}$ is an ordinary category, then naturally $\mathbf{Cat}(N(\mathcal{C})) \cong N(\mathbf{Cat}(\mathcal{C}))$.* □

Proposition 3.1.8. *Let $\mathcal{C}$ be a finitely complete ∞-category and $X_\bullet \colon N(\Delta)^{op} \to \mathcal{C}$ a simplicial object in $\mathcal{C}$. The following conditions are equivalent:*

(1) $X_\bullet$ is a category object.

(2) For any $n \geq 2$ and $0 \leq i \leq n$, the inclusion $\Delta^{\{0,\ldots,i\}} \cup \Delta^{\{i,\ldots,n\}} \hookrightarrow \Delta^n$ induces an equivalence $X(\Delta^n) \xrightarrow{\sim} X\left(\Delta^{\{0,\ldots,i\}} \cup \Delta^{\{i,\ldots,n\}}\right)$.

(3) For any $n \geq 2$ and $0 < i < n$, the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ induces an equivalence $X(\Delta^n) \xrightarrow{\sim} X(\Lambda^n_i)$.

(4) Any inner-anodyne map $f \colon K \hookrightarrow K'$ of finite simplicial sets induces an equivalence $X(K') \xrightarrow{\sim} X(K)$.

[2] In the sense of [Lan21, Definition 4.4.2]. This terminology is not consistent across the literature.

Proof. This is a variation of [Lur09b, Proposition 6.1.2.6]. □

Remark 3.1.9. The condition (2) in this Proposition is equivalent to the square

$$\begin{array}{ccc} X_n & \longrightarrow & X_{\{i,\dots,n\}} \\ \downarrow & & \downarrow \\ X_{\{0,\dots,i\}} & \longrightarrow & X_{\{i\}} \end{array} \tag{2}$$

induced by $X_\bullet$ being a pullback square for all $n \geq 2$ and $0 \leq i \leq n$. This follows by combining the canonical equivalence $X(\Delta^n) \simeq X_n$ for $n \geq 0$ (induced by the final object $\mathrm{id}_{\Delta^n} \in \Delta^{nd}_{/\Delta^n}$) and the decomposition $\Delta^{nd}_{/\Delta^{\{0,\dots,i\}} \cup \Delta^{\{i,\dots,n\}}} \cong \Delta^{nd}_{/\Delta^{\{0,\dots,i\}}} \sqcup_{\Delta_{/\Delta^{\{i\}}}} \Delta^{nd}_{/\Delta^{\{i,\dots,n\}}}$, using [Lur09b, Proposition 4.4.2.2].

Definition 3.1.10. A *groupoid object* in a finitely complete ∞-category $\mathcal{C}$ is a simplicial object

$$X_\bullet \colon N(\Delta)^{op} \to \mathcal{C}$$

such that the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ induces an equivalence $X(\Delta^n) \xrightarrow{\sim} X(\Lambda^n_i)$ for any $n \geq 0$ and $0 \leq i \leq n$. The ∞-category $\mathbf{Gpd}(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}(N(\Delta)^{op}, \mathcal{C})$ on the groupoid objects of $\mathcal{C}$. In particular, $\mathbf{Gpd}(\mathcal{C}) \subseteq \mathbf{Cat}(\mathcal{C})$.

Proposition 3.1.11. *A finitely continuous functor $f \colon \mathcal{C} \to \mathcal{D}$ of finitely complete ∞-categories induces a functor $f_* \colon \mathbf{Cat}(\mathcal{C}) \to \mathbf{Cat}(\mathcal{D})$ resp. a functor $f_* \colon \mathbf{Gpd}(\mathcal{C}) \to \mathbf{Gpd}(\mathcal{D})$.*

Proof. The post-composition functor $f_* \colon \mathrm{Fun}(N(\Delta)^{op}, \mathcal{C}) \to \mathrm{Fun}(N(\Delta)^{op}, \mathcal{D})$ restricts to the desired functors since f preserves pullbacks, using Remark 3.1.9 resp. [Lur09b, Proposition 6.1.2.6, (4")]. □

This next Proposition is implicit in [Lur09a], but we present an alternative proof.

Proposition 3.1.12. *If $\mathcal{C}$ is a presentable ∞-category, then $\mathbf{Cat}(\mathcal{C}) \subseteq \mathrm{Fun}(N(\Delta)^{op}, \mathcal{C})$ is a strongly reflective subcategory. In particular, $\mathbf{Cat}(\mathcal{C})$ is presentable.*

Proof. There is a small ∞-category $\mathcal{D}$ and a fully faithful functor $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{D})$ whose essential image is a strongly reflective subcategory ([Lur09b, Theorem 5.5.1.1]). Then, we obtain a diagram

$$\begin{array}{ccccc} \mathrm{Fun}(N(\Delta)^{op}, \mathcal{C}) & \hookrightarrow & \mathrm{Fun}(N(\Delta)^{op}, \mathcal{P}(\mathcal{D})) & \xrightarrow{\sim} & \mathrm{Fun}(\mathcal{D}^{op}, \mathrm{Fun}(N(\Delta)^{op}, \mathbf{Spc})) \\ \cup\mathrm{l} & & \cup\mathrm{l} & & \cup\mathrm{l} \\ \mathbf{Cat}(\mathcal{C}) & \hookrightarrow & \mathbf{Cat}(\mathcal{P}(\mathcal{D})) & \xrightarrow{\sim} & \mathrm{Fun}(\mathcal{D}^{op}, \mathbf{Cat}(\mathbf{Spc})). \end{array}$$

The left square is a (strict) pullback square since the fully faithful right adjoint $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{D})$ preserves and reflects the pullback squares of Remark 3.1.9. The categories in the upper row are presentable by [Lur09b, Proposition 5.5.3.6]. Thus, by [Lur09b, Lemma 5.5.4.17], it suffices to verify

that $\mathrm{Fun}(\mathcal{D}^{op}, \mathbf{Cat}(\mathbf{Spc})) \subseteq \mathrm{Fun}\left(\mathcal{D}^{op}, \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)\right)$ is a strongly reflective subcategory. If $\mathbf{Cat}(\mathbf{Spc}) \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)$ is a strongly reflective subcategory, this follows from [Lan21, Proposition 5.1.16]. This holds by [Lur09b, Proposition 5.5.4.15] since it is the full subcategory of S-local objects for $S = \{\mathrm{Sp}^n \hookrightarrow \Delta^n | n \geq 2\}$ (or rather its image under $N(\mathbf{sSet}) \hookrightarrow \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)$). □

Definition 3.1.13. A simplicial object $X_\bullet \colon N\left(\Delta\right)^{op} \to \mathcal{C}$ in a finitely completely ∞-category $\mathcal{C}$ is *constant* if it is contained in the essential image of the diagonal $\delta \colon \mathcal{C} \to \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{C}\right)$.

Proposition 3.1.14. *Let K be a weakly contractible simplicial set (e.g. $K = N\left(\Delta\right)^{op}$) and $\mathcal{C}$ an ∞-category. The diagonal $\delta \colon \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ is fully faithful and its essential image consists precisely of those diagrams $X \colon K \to \mathcal{C}$ whose image is contained in $\mathcal{C}^{\simeq}$.*

Proof. For any $x, y \in \mathcal{C}$, the induced map $\mathrm{map}_{\mathcal{C}}(x, y) \to \mathrm{map}_{\mathrm{Fun}(K,\mathcal{C})}\left(\delta(x), \delta(y)\right) \cong \mathrm{Fun}(K, \mathrm{map}_{\mathcal{C}}(x, y))$ is a homotopy equivalence since $\mathrm{map}_{\mathcal{C}}(x, y)$ is a Kan complex and K is weakly contractible. The image of any constant map is clearly contained in $\mathcal{C}^{\simeq}$, which is closed under equivalences, so this also holds for any map in the essential image and, conversely, any map $K \to \mathcal{C}$ factoring through $\mathcal{C}^{\simeq}$ is naturally equivalent to a constant map since K is weakly contractible and $\mathcal{C}^{\simeq}$ is a Kan complex. □

Proposition 3.1.15 ([Lur09a, Proposition 1.1.14])**.** *Let $\mathcal{C}$ be a finitely complete ∞-category. The inclusion $\mathbf{Gpd}(\mathcal{C}) \subseteq \mathbf{Cat}(\mathcal{C})$ admits a right adjoint $(-)^{\sim} \colon \mathbf{Cat}(\mathcal{C}) \to \mathbf{Gpd}(\mathcal{C})$.* □

Remark 3.1.16. The adjunction is essentially described by $X_0^{\sim} \simeq X_0$ and $X_1^{\sim} \simeq X(K) \to X(\Delta^1) \simeq X_1$, where $K = \Delta^3 \sqcup_{\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}}} \left(\Delta^0 \sqcup \Delta^0\right)$.

Now that we have developed internal categories (and groupoids), we want to establish a variant of 'enriched categories' by generalizing the discreteness condition discussed in Subsection 2.1.

Definition 3.1.17 ([Lur09a, Definition 1.2.1])**.** A *distributor* is a presentable ∞-category $\mathcal{Y}$ together with a presentable full subcategory $\mathcal{X}$ that is stable under small limits and colimits (in particular, stable under equivalences) satisfying:

(1) For any morphism $f \colon y \to x$ in $\mathcal{Y}$, where $x \in \mathcal{X}$, the pullback functor $f^* \colon \mathcal{X}_{/x} \to \mathcal{Y}_{/y}$ preserves small colimits.

(2) The functor $\mathcal{X} \to \mathbf{Cat}_\infty^{op}$ classifying the cartesian fibration $\mathrm{Fun}\left(\Delta^1, \mathcal{Y}\right) \times_{\mathrm{Fun}(\{1\},\mathcal{Y})} \mathcal{X} \to \mathcal{X}$ preserves small colimits.

Remark 3.1.18. The adjoint functor theorem implies that the inclusion $\mathcal{X} \subseteq \mathcal{Y}$ admits a left adjoint $\tau \colon \mathcal{Y} \to \mathcal{X}$, called *truncation*, and a right adjoint $\kappa \colon \mathcal{Y} \to \mathcal{X}$, called *core*.

Remark 3.1.19. The identity $\mathcal{X} \subseteq \mathcal{X}$ is a distributor if and only if $\mathcal{X}$ is an ∞-topos ([Lur09a, Example 1.2.3]). If $\mathcal{X} \subseteq \mathcal{Y}$ is a distributor, then $\mathcal{X} \subseteq \mathcal{X}$ is a distributor ([Lur09a, Remark 1.2.6]).

Definition 3.1.20 ([Lur09a, Definition 1.3.3])**.** An *absolute distributor* is a presentable ∞-category $\mathcal{Y}$ such that the (essentially unique) colimit- and final object-preserving functor $\mathbf{Spc} \to \mathcal{Y}$ ([Lur09b, Theorem 5.1.5.6]) is fully faithful and its essential image $\mathcal{X} \subseteq \mathcal{Y}$ is a distributor.

Definition 3.1.21 ([Lur09a, Definition 1.2.7])**.** Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. A *Segal space object* in this distributor is a category object $Y_\bullet \in \mathbf{Cat}(\mathcal{Y})$ such that $Y_0 \in \mathcal{X}$. The ∞-category $SS_{\mathcal{X}\subseteq\mathcal{Y}}$ is the full subcategory of $\mathbf{Cat}(\mathcal{Y})$ on the Segal space objects.

Proposition 3.1.22 ([Lur09a, Remark 1.2.8])**.** *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. Then, $SS_{\mathcal{X}\subseteq\mathcal{Y}}$ is a strongly reflective subcategory of $\mathbf{Cat}(\mathcal{Y})$. In particular, $SS_{\mathcal{X}\subseteq\mathcal{Y}}$ is a presentable ∞-category.*

Proof. There is a strict pullback diagram of ∞-categories

$$\begin{array}{ccc} SS_{\mathcal{X}\subseteq\mathcal{Y}} & \subseteq & \mathbf{Cat}(\mathcal{Y}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathcal{X} & \subseteq & \mathcal{Y}. \end{array} \tag{3}$$

The ∞-category $\mathcal{Y}$ is presentable, hence $\mathbf{Cat}(\mathcal{Y})$ is presentable by Proposition 3.1.12. The right vertical map is a right adjoint (its left adjoint is the diagonal $\delta\colon \mathcal{Y} \to \mathbf{Cat}(\mathcal{Y})$) and $\mathcal{X} \subseteq \mathcal{Y}$ is a strongly reflective subcategory (Remark 3.1.18), so the claim follows by [Lur09b, Lemma 5.5.4.17]. □

For later use, we also record

Proposition 3.1.23. *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. Then, $SS_{\mathcal{X}\subseteq\mathcal{Y}} \subseteq \mathbf{Cat}(\mathcal{Y})$ is a coreflective subcategory.*

Proof. The strict pullback square (3) is a pullback square in $\mathbf{Cat}_\infty$ since $\mathcal{X} \subseteq \mathcal{Y}$ is an isofibration. The cospan $\mathcal{X} \subseteq \mathcal{Y} \leftarrow \mathbf{Cat}(\mathcal{Y})$ is contained in $\mathrm{Pr}^L \subseteq \mathbf{Cat}_\infty$. Indeed, ev_0 admits a right adjoint, the right Kan extension T along $\{0\} \subseteq N(\Delta)^{op}$, by [Lur09b, Proposition 4.3.2.17], which maps an object $Y \in \mathcal{Y}$ to the simplicial object $[n] \mapsto Y^{n+1}$ and this is a category object, as desired. Thus, the claim follows by [Lur09b, Proposition 5.5.3.13]. □

The inclusion $\mathbf{Gpd}(\mathcal{X}) \subseteq \mathbf{Cat}(\mathcal{X}) \subseteq \mathbf{Cat}(\mathcal{Y})$ admits a right adjoint given by composing the right adjoints $\kappa_*\colon \mathbf{Cat}(\mathcal{Y}) \to \mathbf{Cat}(\mathcal{X})$ (compare Proposition 3.1.11) and $(-)^\sim\colon \mathbf{Cat}(\mathcal{X}) \to \mathbf{Gpd}(\mathcal{X})$ of Proposition 3.1.15. This restricts to a right adjoint $\mathrm{Gp}\colon SS_{\mathcal{X}\subseteq\mathcal{Y}} \to \mathbf{Gpd}(\mathcal{X})$ to the inclusion $\mathbf{Gpd}(\mathcal{X}) \subseteq \mathbf{Cat}(\mathcal{X}) \subseteq SS_{\mathcal{X}\subseteq\mathcal{Y}}$.

Definition 3.1.24. Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. A Segal space object $Y_\bullet$ is *complete* if the groupoid object $\mathrm{Gp}\, Y_\bullet \in \mathbf{Gpd}(\mathcal{X})$ is constant. The ∞-category $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is the full subcategory of $SS_{\mathcal{X}\subseteq\mathcal{Y}}$ on the complete Segal space objects. For an absolute distributor $\mathcal{Y}$, it is denoted $CSS_{\mathcal{Y}}$.

Remark 3.1.25. To motivate this, let us consider the case of an absolute distributor. The ∞-category **Spc** fulfills a role analogous to that of **Set** in ordinary category theory, so this is a generalization of the setting in Subsection 2.1. Now, take a (complete) Segal space object $Y_\bullet$, which we think of as a $\mathcal{Y}$-enriched category. Our intuition demands that we consider $Y_0 \in \mathbf{Spc}$ to be its underlying 'space of objects'. The completeness condition arises from the discrepancy that **Spc** *also* fulfills a role analogous to **Gpd** in ordinary category theory. Indeed, we ought to consider $\kappa_* Y_\bullet$ the underlying **Spc**-enriched category and such a gadget should model an ∞-category (this will be made precise by Theorem 3.2.4), so $\mathrm{Gp}\, Y_\bullet = (\kappa_* Y_\bullet)^\sim$ is the 'underlying ∞-groupoid'. However, the 'space of objects' and 'underlying ∞-groupoid' should morally be the same thing and this is what completeness ensures.

Definition 3.1.26. A map of Segal space objects $f_\bullet \colon Y_\bullet \to Y'_\bullet$ is a *Segal equivalence* if it satisfies:

(1) The induced map $|\mathrm{Gp}\, Y_\bullet| \to |\mathrm{Gp}\, Y'_\bullet|$ is an equivalence in $\mathcal{X}$.

(2) The induced square

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y'_1 \\ \downarrow & & \downarrow \\ Y_0 \times Y_0 & \longrightarrow & Y'_0 \times Y'_0 \end{array}$$

is a pullback square in $\mathcal{Y}$.

Remark 3.1.27. A functor $f \colon \mathcal{C} \to \mathcal{D}$ of ∞-categories is fully faithful if and only if the square

$$\begin{array}{ccc} \mathrm{Fun}\left(\Delta^1, \mathcal{C}\right)^\simeq & \longrightarrow & \mathrm{Fun}\left(\Delta^1, \mathcal{D}\right)^\simeq \\ \downarrow & & \downarrow \\ \mathcal{C}^\simeq \times \mathcal{C}^\simeq & \longrightarrow & \mathcal{D}^\simeq \times \mathcal{D}^\simeq \end{array}$$

is a pullback square in **Spc**. Thus, condition (2) should be understood as an analogue of full faithfulness. In case $\mathcal{Y}$ is an absolute distributor, it also admits a pointwise description, though we won't prove this here. Furthermore, if condition (2) is satisfied, then condition (1) is equivalent to the induced map $\pi_0(|\mathrm{Gp}\, Y_\bullet|) \to \pi_0(|\mathrm{Gp}\, Y'_\bullet|)$ being a surjection (combine [Lur09a, Remark 1.2.17] and [Lur09b, Corollary 7.2.1.15]). In light of the previous discussion, this condition should be understood as an analogue of essential surjectivity.

Theorem 3.1.28 ([Lur09a, Theorem 1.2.13])**.** *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. Then, $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is a strongly reflective subcategory of $SS_{\mathcal{X}\subseteq\mathcal{Y}}$, obtained by localizing at the Segal equivalences. In particular, $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is a presentable ∞-category and the inclusion has a left adjoint $L \colon SS_{\mathcal{X}\subseteq\mathcal{Y}} \to CSS_{\mathcal{X}\subseteq\mathcal{Y}}$.* □

Remark 3.1.29. The colimits in $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ are mysterious, but limits are straightforward — they are computed pointwise. Indeed, this is true in $\mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{Y}\right)$ and $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is stable under limits in $\mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{Y}\right)$, either since the inclusion is a right adjoint or by direct inspection.

The central result that allows us to iterate this construction is

Proposition 3.1.30 ([Lur09a, Proposition 1.3.2])**.** *Let $\mathcal{X}\subseteq\mathcal{Y}$ be a distributor, $\mathcal{X}'$ the essential image of the diagonal $\delta\colon\mathcal{X}\to CSS_{\mathcal{X}\subseteq\mathcal{Y}}\subseteq\mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{Y}\right)$. Then, $\mathcal{X}'\subseteq CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is a distributor.* □

Remark 3.1.31. The truncation and core of the induced distributor $\mathcal{X}\simeq\mathcal{X}'\subseteq CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ are given by $\tau\left|-\right|:CSS_{\mathcal{X}\subseteq\mathcal{Y}}\to\mathcal{X}\simeq\mathcal{X}'$, where $\left|-\right|=\mathrm{colim}_{N(\Delta)^{op}}$, and $\mathrm{ev}_0\colon CSS_{\mathcal{X}\subseteq\mathcal{Y}}\to\mathcal{X}\simeq\mathcal{X}'$ respectively.

Now, we study some more properties of these constructions. Following [Gol24, Definition 3.3.1], we call a functor $f\colon\mathcal{C}\to\mathcal{D}$ *relatively left exact* with respect to a full subcategory $\mathcal{C}_0\subseteq\mathcal{C}$ that is closed under finite limits if f preserves pullbacks over $\mathcal{C}_0$ and $f|_{\mathcal{C}_0}$ is left exact, i.e. preserves finite limits (this is automatic if f is a Bousfield localization and $\mathcal{C}_0\subseteq\mathcal{D}$). In particular, f preserves finite products.

Lemma 3.1.32. *Let $\mathcal{X}$ be an ∞-topos. The functor $\mathrm{colim}_{N(\Delta)^{op}}\colon\mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{X}\right)\to\mathcal{X}$ is left exact relative to $\mathcal{X}'$, the essential image of the diagonal $\delta\colon\mathcal{X}\to\mathrm{Fun}(K,\mathcal{X})$.*

Proof. Since $N\left(\Delta\right)^{op}$ is weakly contractible, $\mathrm{colim}_{N(\Delta)^{op}}\delta\left(-\right)\simeq\mathrm{id}_{\mathcal{X}}$ by [Lur09b, Corollary 4.4.4.10], hence $\mathrm{colim}_{N(\Delta)^{op}}|_{\mathcal{X}'}$ is an equivalence and in particular left exact. Let $Y_\bullet,Y'_\bullet\in\mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{X}\right)$, $x\in\mathcal{X}$ and a cospan $Y_\bullet\to\delta(x)\leftarrow Y'_\bullet$ be given. Then, we calculate

$$\begin{aligned}\operatorname*{colim}_{[n]\in N(\Delta)^{op}}(Y_n\times_x Y'_n)&\simeq\operatorname*{colim}_{([n],[m])\in N(\Delta)^{op}\times N(\Delta)^{op}}(Y_n\times_x Y'_m)\simeq\operatorname*{colim}_{[n]\in N(\Delta)^{op}}\operatorname*{colim}_{[m]\in N(\Delta)^{op}}(Y_n\times_x Y'_m)\\&\simeq\operatorname*{colim}_{[n]\in N(\Delta)^{op}}\left(Y_n\times_x\left(\operatorname*{colim}_{[m]\in N(\Delta)^{op}}Y'_m\right)\right)\simeq\left(\operatorname*{colim}_{[n]\in N(\Delta)^{op}}Y_n\right)\times_x\left(\operatorname*{colim}_{[m]\in N(\Delta)^{op}}Y'_m\right).\end{aligned}$$

Here, we have used that $N\left(\Delta\right)^{op}$ is sifted ([Lur09b, Lemma 5.5.8.4]), that colimits commute with colimits and that colimits are universal in $\mathcal{X}$. □

Lemma 3.1.33. *Let $\mathcal{X}\subseteq\mathcal{Y}$ be a distributor. The collection of Segal equivalences in $SS_{\mathcal{X}\subseteq\mathcal{Y}}$ is stable under pullbacks over $\mathcal{X}'$.*

Proof. The condition (1) of Definition 3.1.26 is stable under pullbacks over $\mathcal{X}'$ as Gp preserves them since it is right adjoint and the geometric realization functor $\left|-\right|$ preserves them by Lemma 3.1.32. The condition (2) is stable under any limits since limits commute with limits. □

Corollary 3.1.34 ([Gol24, Proposition 3.5.3])**.** *Let $\mathcal{X}\subseteq\mathcal{Y}$ be a distributor. The completion functor $L\colon SS_{\mathcal{X}\subseteq\mathcal{Y}}\to CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ of Theorem 3.1.28 is left exact relative to $\mathcal{X}'$.*

Proof. The restriction $L|_{\mathcal{X}'}$ is naturally equivalent to the inclusion $\mathcal{X}' \subseteq CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ since constant Segal spaces are complete. For a cospan $Y_\bullet \to X_\bullet \leftarrow Y_\bullet$ in $SS_{\mathcal{X}\subseteq\mathcal{Y}}$ with $X_\bullet \in \mathcal{X}'$, the natural map $Y_\bullet \times_{X_\bullet} Y'_\bullet \to LY_\bullet \times_{X_\bullet} LY'_\bullet$ is a Segal equivalence by Lemma 3.1.33, hence the induced map $L(Y_\bullet \times_{X_\bullet} Y'_\bullet) \to LY_\bullet \times_{X_\bullet} LY'_\bullet$ is an equivalence by Theorem 3.1.28. □

Proposition 3.1.35 ([Gol24, Lemma 3.3.2])**.** *Let $\mathcal{X}_0 \subseteq \mathcal{Y}_0$ and $\mathcal{X}_1 \subseteq \mathcal{Y}_1$ be distributors, and $F\colon \mathcal{Y}_0 \to \mathcal{Y}_1$ a functor such that*

(1) F preserves pullbacks over objects of $\mathcal{X}_0$.

(2) F restricts to a geometric morphism $f\colon \mathcal{X}_0 \to \mathcal{X}_1$ of ∞-topoi,

(3) F is compatible with cores, i.e. $\kappa_1 F \simeq f\kappa_0$, and

Then, F induces a functor $F_*\colon CSS_{\mathcal{X}_0\subseteq\mathcal{Y}_0} \to CSS_{\mathcal{X}_1\subseteq\mathcal{Y}_1}$, which again satisfies these hypotheses relative to the induced distributors $\mathcal{X}'_0 \subseteq CSS_{\mathcal{X}_0\subseteq\mathcal{Y}_0}$ and $\mathcal{X}'_1 \subseteq CSS_{\mathcal{X}_1\subseteq\mathcal{Y}_1}$ of Proposition 3.1.30.

Proof. The post-composition functor $F_*\colon \mathrm{Fun}(N(\Delta)^{op}, \mathcal{Y}_0) \to \mathrm{Fun}(N(\Delta)^{op}, \mathcal{Y}_1)$ restricts to a functor $F_*\colon SS_{\mathcal{X}_0\subseteq\mathcal{Y}_0} \to SS_{\mathcal{X}_1\subseteq\mathcal{Y}_1}$, since F preserves pullbacks over objects of $\mathcal{X}_0$ (compare Proposition 3.1.11) and $F\mathcal{X}_0 \subseteq \mathcal{X}_1$. There is an adjunction $g\colon \mathcal{X}_1 \rightleftarrows \mathcal{X}_0\colon f$, where g is finitely continuous. This induces an adjunction $g_*\colon \mathrm{Fun}(N(\Delta)^{op}, \mathcal{X}_1) \rightleftarrows \mathrm{Fun}(N(\Delta)^{op}, \mathcal{X}_0) : f_*$, which restricts to adjunctions $g_*\colon \mathbf{Cat}(\mathcal{X}_1) \rightleftarrows \mathbf{Cat}(\mathcal{X}_0)\colon f_*$ and $g_*\colon \mathbf{Gpd}(\mathcal{X}_1) \rightleftarrows \mathbf{Gpd}(\mathcal{X}_0)\colon f_*$ by Proposition 3.1.11. The uniqueness of right adjoints now implies that $(f_*(-))^\sim \simeq f_*((-)^\sim)$. Thus, given a complete Segal space object $Y_\bullet$ in $\mathcal{X}_0 \subseteq \mathcal{Y}_0$, we have

$$\mathrm{Gp}_1(F_*Y_\bullet) \simeq ((\kappa_1 F)_*Y_\bullet)^\sim \simeq ((f\kappa_0)_*Y_\bullet)^\sim \simeq f_*((\kappa_0)_*Y_\bullet)^\sim \simeq f_*(\mathrm{Gp}_0 Y_\bullet),$$

which is constant, hence $F_*Y_\bullet$ is a complete Segal space object in $\mathcal{X}_1 \subseteq \mathcal{Y}_1$, as desired.

The functor $F_*\colon CSS_{\mathcal{X}_0\subseteq\mathcal{Y}_0} \to CSS_{\mathcal{X}_1\subseteq\mathcal{Y}_1}$ is given pointwise by F. Thus, it restricts to a functor $\mathcal{X}'_0 \to \mathcal{X}'_1$ on constant objects that is equivalent to f, so also a geometric morphism of ∞-topoi, it commutes with the evaluations at 0, which are the induced core functors by Remark 3.1.31, and preserves pullbacks over objects of $\mathcal{X}'_0$ since pullbacks of complete Segal space objects are computed pointwise by Remark 3.1.29. It follows that F_* again satisfies the hypotheses. □

Proposition 3.1.36. *Let $\mathcal{X}_0 \subseteq \mathcal{Y}_0$ and $\mathcal{X}_1 \subseteq \mathcal{Y}_1$ be distributors, and $F\colon \mathcal{Y}_0 \rightleftarrows \mathcal{Y}_1\colon G$ an adjunction that restricts to an adjunction $f\colon \mathcal{X}_0 \rightleftarrows \mathcal{X}_1\colon g$.*

(1) If F is left exact relative to $\mathcal{X}_0$, this induces an adjunction $LF_*\colon CSS_{\mathcal{X}_0\subseteq\mathcal{Y}_0} \rightleftarrows CSS_{\mathcal{X}_1\subseteq\mathcal{Y}_1}\colon G_*$.

(2) If F is subject to 3.1.35, this induces an adjunction $F_*\colon CSS_{\mathcal{X}_0\subseteq\mathcal{Y}_0} \rightleftarrows CSS_{\mathcal{X}_1\subseteq\mathcal{Y}_1}\colon G_*$.

Furthermore, these adjunctions again satisfy the respective hypotheses relative to the induced distributors $\mathcal{X}_0' \subseteq CSS_{\mathcal{X}_0 \subseteq \mathcal{Y}_0}$ and $\mathcal{X}_1' \subseteq CSS_{\mathcal{X}_1 \subseteq \mathcal{Y}_1}$ of Proposition 3.1.30.

Proof. In either case, G satisfies the hypotheses of Proposition 3.1.35 and hence G_* is defined. Indeed, G preserves pullbacks as it is a right adjoint, its restricts g is a geometric morphism since its left adjoint f is either left exact by assumption or since it is a right adjoint itself, and $\kappa_0 G \simeq g\kappa_1$ holds since these are right adjoints to $F|_{\mathcal{X}_0} \simeq f$. This immediately implies (2).

Now, assume (1). The assumptions imply that there is an induced adjunction $F_* \colon SS_{\mathcal{X}_0 \subseteq \mathcal{Y}_0} \rightleftarrows SS_{\mathcal{X}_1 \subseteq \mathcal{Y}_1} \colon G_*$, as in Proposition 3.1.35, hence an adjunction $LF_* \colon SS_{\mathcal{X}_0 \subseteq \mathcal{Y}_0} \rightleftarrows CSS_{\mathcal{X}_1 \subseteq \mathcal{Y}_1} \colon G_*$. Since G_* preserves complete Segal space objects, the desired adjunction is obtained by restriction.

To see that these constructions can be iterated, note that either adjunction restricts to an adjunction $\mathcal{X}_0' \rightleftarrows \mathcal{X}_1'$ that is equivalent to $f \colon \mathcal{X}_0 \rightleftarrows \mathcal{X}_1 \colon g$ (in case (1), note that constant Segal spaces are complete). The remaining properties are obtained by the arguments in Proposition 3.1.35 and Corollary 3.1.34. □

Remark 3.1.37. In case (2), the uniqueness of left adjoints implies that canonically $L_1 F_* \simeq F_* L_0$.

Proposition 3.1.38. *In the situation of Proposition 3.1.35, if F is an isofibration, then F_* is an isofibration.*

Proof. The functor $F_* \colon \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}_0\right) \to \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}_1\right)$ is an isofibration by [Lan21, Proposition 2.2.5] and the full subcategory $CSS_{\mathcal{X}_0 \subseteq \mathcal{Y}_0}$ is stable under equivalences in $\mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}_0\right)$, hence the restriction $F_* \colon CSS_{\mathcal{X}_0 \subseteq \mathcal{Y}_0} \to CSS_{\mathcal{X}_1 \subseteq \mathcal{Y}_1}$ is an isofibration. □

Proposition 3.1.39. *In the situation of Proposition 3.1.35, if F is fully faithful, then F_* is fully faithful.*

Proof. The functor $F_* \colon \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}_0\right) \to \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}_1\right)$ is fully faithful by [Lan21, Lemma 3.3.15], hence so is its restriction $F_* \colon CSS_{\mathcal{X}_0 \subseteq \mathcal{Y}_0} \to CSS_{\mathcal{X}_1 \subseteq \mathcal{Y}_1}$ to full subcategories. □

It turns out that many properties are stable under the construction $(\mathcal{X} \subseteq \mathcal{Y}) \mapsto (\mathcal{X}' \subseteq CSS_{\mathcal{X} \subseteq \mathcal{Y}})$. This includes exactness of filtered colimits in $\mathcal{Y}$ and $\kappa \colon \mathcal{Y} \to \mathcal{X}$ preserving filtered colimits ([Lur09a, Proposition 1.2.29]), $\mathcal{Y}$ being extensive and $\kappa \colon \mathcal{Y} \to \mathcal{X}$ preserving small coproducts (this will not be proven here) and $\mathcal{Y}$ being cartesian closed and $\mathcal{X}$ an exponential ideal in $\mathcal{Y}$, as we shall now prove. The basic theory of cartesian closed ∞-categories is outlined in Appendix A.

Proposition 3.1.40. *Let $\mathcal{C}$ be a cartesian closed, presentable ∞-category. Then, $\mathbf{Cat}(\mathcal{C})$ is an exponential ideal in $\mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{C}\right)$.*

Proof. There is a small ∞-category $\mathcal{D}$ and a fully faithful functor $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{D})$ whose essential image is a strongly reflective exponential ideal by Proposition A.0.16. Then, we obtain a diagram

$$\begin{array}{ccccc} \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{C}\right) & \hookrightarrow & \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{P}(\mathcal{D})\right) & \xrightarrow{\sim} & \mathrm{Fun}\left(\mathcal{D}^{op}, \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)\right) \\ \cup\mathrm{I} & & \cup\mathrm{I} & & \cup\mathrm{I} \\ \mathbf{Cat}(\mathcal{C}) & \hookrightarrow & \mathbf{Cat}(\mathcal{P}(\mathcal{D})) & \xrightarrow{\sim} & \mathrm{Fun}(\mathcal{D}^{op}, \mathbf{Cat}(\mathbf{Spc})). \end{array}$$

The left square is a (strict) pullback (see Proposition 3.1.12). The top row is a cartesian closed functor by Proposition A.0.3 and Corollary A.0.13, so Proposition A.0.17 reduces this to verifying that $\mathrm{Fun}(\mathcal{D}^{op}, \mathbf{Cat}(\mathbf{Spc})) \subseteq \mathrm{Fun}\left(\mathcal{D}^{op}, \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)\right)$ is an exponential ideal. This follows by Corollary A.0.13, if $\mathbf{Cat}(\mathbf{Spc}) \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)$ is an exponential ideal.

Now, $\mathbf{Cat}(\mathbf{Spc}) \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)$ is the full subcategory of S-local objects for S the (essentially) small collection of inner-anodyne maps between finite simplicial sets (or rather its image under $N(\mathbf{sSet}) \hookrightarrow \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)$) by Proposition 3.1.8. This collection is closed under taking products with id_K for K a finite simplicial set by [Lan21, Lemma 1.3.31], in particular for $K = \Delta^n$, $n \geq 0$. These generate $\mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right)$ under colimits by [Lur09b, Corollary 5.1.5.8], hence the claim follows by Corollary A.0.10. □

Theorem 3.1.41. *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor such that $\mathcal{Y}$ is cartesian closed and $\mathcal{X} \subseteq \mathcal{Y}$ an exponential ideal. Then, $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is cartesian closed and $\mathcal{X}' \subseteq CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ an exponential ideal.*

Proof. By Proposition 3.1.40, $\mathbf{Cat}(\mathcal{Y}) \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}\right)$ is an exponential ideal, hence $\mathbf{Cat}(\mathcal{Y})$ is cartesian closed by Proposition A.0.3. The full subcategory $SS_{\mathcal{X}\subseteq\mathcal{Y}} \subseteq \mathbf{Cat}(\mathcal{Y})$ is coreflective by Proposition 3.1.23, hence cartesian closed by Proposition A.0.7. Lastly, the full subcategory $CSS_{\mathcal{X}\subseteq\mathcal{Y}} \subseteq SS_{\mathcal{X}\subseteq\mathcal{Y}}$ is an exponential ideal by characterization (2) of Proposition A.0.9 and Corollary 3.1.34. In particular, $CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is cartesian closed by Proposition A.0.3.

Then, by Proposition A.0.11, $\mathcal{X}' \subseteq CSS_{\mathcal{X}\subseteq\mathcal{Y}}$ is an exponential ideal if and only if $\mathcal{X}' \subseteq SS_{\mathcal{X}\subseteq\mathcal{Y}}$ is an exponential ideal. This follows if $\mathcal{X}' \subseteq \mathbf{Cat}\left(\mathcal{Y}\right)$ is an exponential ideal by Proposition A.0.7, which is the case if and only if $\mathcal{X}' \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}\right)$ is an exponential ideal by Proposition A.0.11 again. This is the case since $\mathcal{X}' \subseteq \mathcal{Y}'$ is an exponential ideal by assumption and $\mathcal{Y}' \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{Y}\right)$ is an exponential ideal by characterization (3) of Proposition A.0.9 and [Lur09b, Lemma 5.5.8.11]. □

Lastly, we discuss a version of the 'Delooping Hypothesis'. Recall that a *monoid object* in an ∞-category $\mathcal{C}$ admitting finite products is a simplicial object $X_\bullet \colon N\left(\Delta\right)^{op} \to \mathcal{C}$ such that, for any $n \geq 0$, the maps $X([n]) \to X(\{i-1, i\})$, $i = 1, \ldots, n$ exhibit X_n as the n-fold product of X_1 ([Lur17, Definition 4.1.2.5]). Equivalently, X satisfies the Segal condition and $X_0 \simeq *$. Intuitively, we interpret this as the object X_1 together with a multiplication law that is unital and associative up to coherent homotopy. The ∞-category $\mathrm{Mon}(\mathcal{C})$ is the full subcategory of $\mathbf{Cat}(\mathcal{C})$ on the monoid objects.

Proposition 3.1.42. *For any ∞-category $\mathcal{C}$ admitting finite products, there is a loop object functor $\Omega\colon \mathbf{Cat}(\mathcal{C})_* \to \mathrm{Mon}(\mathcal{C})_* \simeq \mathrm{Mon}(\mathcal{C})$, taking a pointed category object $* \to Y_\bullet$ to the monoid object $[n] \mapsto Y_n \times_{Y_0^{\times(n+1)}} \{*\}$, which is right adjoint to the inclusion $\mathrm{Mon}(\mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C})_* \hookrightarrow \mathbf{Cat}(\mathcal{C})_*$.*

Remark 3.1.43. Intuitively, the pointing $* \to Y_\bullet$ gives an object in the 'internal category' $Y_\bullet$ and $(\Omega Y_\bullet)_1 = Y_1 \times_{Y_0^{\times 2}} \{*\}$ is its object of endomorphisms, whose monoid structure is given by composition.

Proof. For any monoid object $X_\bullet$, we have $X_0 \simeq *$, hence

$$\mathrm{map}_{\mathrm{Mon}(\mathcal{C})}(*, X_\bullet) \simeq \mathrm{map}_{\mathcal{C}}(*, X_0) \simeq \mathrm{map}_{\mathcal{C}}(*, *) \simeq *.$$

Thus, $*$ is also an initial object in $\mathrm{Mon}(\mathcal{C})$, so the projection $\mathrm{Mon}(\mathcal{C})_* \to \mathrm{Mon}(\mathcal{C})$ that forgets the pointing is an equivalence of ∞-categories.

To describe the loop object functor Ω, let $T\colon \mathcal{C}_* \to \mathbf{Cat}(\mathcal{C})_*$ denote the map induced by right Kan extension along $[0] \hookrightarrow N(\Delta)^{op}$, i.e. the functor taking a pointed object X to the pointed category object $[n] \mapsto X^{\times(n+1)}$. The pointing $* \to Y_\bullet$ induces a cospan $Y_\bullet \to T(Y_0) \leftarrow T(*) \simeq *$ in $\mathbf{Cat}(\mathcal{C})_*$, whose pullback is the pointed category object $\Omega Y_\bullet$. The forgetful functor $\mathbf{Cat}(\mathcal{C})_* \to \mathbf{Cat}(\mathcal{C})$ preserves small limits ([Lur09b, Proposition 1.2.13.8]) and $\mathbf{Cat}(\mathcal{C}) \subseteq \mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathcal{C}\right)$ is closed under small limits, so we can compute $(\Omega Y_\bullet)_0 \simeq \lim\left(Y_0 \to Y_0 \leftarrow *\right) \simeq *$, i.e. $\Omega Y_\bullet$ is a monoid object.

The canonical projection yields a unit map $\Omega Y_\bullet \to Y_\bullet$. Indeed, for any monoid object $X_\bullet$ (together with its essentially unique pointing), there is a pullback square ([Lur09b, Proposition 5.1.3.2])

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{Mon}(\mathcal{C})_*}\left(X_\bullet, \Omega Y_\bullet\right) & \longrightarrow & \mathrm{map}_{\mathbf{Cat}(\mathcal{C})_*}\left(X_\bullet, Y_\bullet\right) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathbf{Cat}(\mathcal{C})_*}\left(X_\bullet, T(*)\right) & \xrightarrow{\sim} & \mathrm{map}_{\mathbf{Cat}(\mathcal{C})_*}\left(X_\bullet, T(Y_0)\right), \end{array}$$

where the bottom horizontal map is an equivalence since either space is contractible. Thus, the top horizontal map is an equivalence and we obtain the desired adjunction. □

Theorem 3.1.44 (Delooping Hypothesis)**.** *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. There is a canonical adjunction*

$$B\colon \mathrm{Mon}(\mathcal{Y}) \rightleftarrows CSS_{\mathcal{X}\subseteq\mathcal{Y},*}\colon \Omega.$$

The left adjoint B is fully faithful, preserves finite products and its essential image consists of those pointed complete Segal space objects $* \to Y_\bullet$ that are connective, i.e. for which $* \to |\mathrm{Gp}\, Y_\bullet| \simeq Y_0$ is an effective epimorphism in $\mathcal{X}$.

Proof. The adjunction of Proposition 3.1.42 restricts to an adjunction $\mathrm{Mon}(\mathcal{Y}) \rightleftarrows SS_{\mathcal{X}\subseteq\mathcal{Y},*}\colon \Omega$. This composes with the adjunction $L\colon SS_{\mathcal{X}\subseteq\mathcal{Y},*} \rightleftarrows CSS_{\mathcal{X}\subseteq\mathcal{Y},*}$, induced on slices by Theorem 3.1.28, to the

desired adjunction. The unit map of this adjunction is given by the composite

$$X_\bullet \xrightarrow{\sim} \Omega X_\bullet \to \Omega L X_\bullet,$$

where the first equivalence is due to Proposition 3.1.42 ([Lan21, Proposition 5.1.8]) and the second map is obtained by applying Ω to the unit map $X_\bullet \to LX_\bullet$, but this unit map is a Segal equivalence by Theorem 3.1.28, hence the square

$$\begin{array}{ccc} X_1 & \longrightarrow & (LX)_1 \\ \downarrow & & \downarrow \\ X_0 \times X_0 & \longrightarrow & (LX)_0 \times (LX)_0 \end{array}$$

is a pullback square and the induced map $(\Omega X)_1 \to (\Omega LX)_1$ on fibers is an equivalence. The forgetful functor $\mathrm{Mon}\,(\mathcal{Y}) \to \mathcal{Y}$ is conservative, so $\Omega X_\bullet \xrightarrow{\sim} \Omega LX_\bullet$ is an equivalence and B is fully faithful by [Lan21, Proposition 5.1.8].

For a monoid object $X_\bullet$, the canonical map $X_\bullet \to LX_\bullet$ is a Segal equivalence by Theorem 3.1.28, hence $* \simeq X_0 \to |\mathrm{Gp}\, LX_\bullet|$ is an effective epimorphism. Conversely, suppose that $* \to Y_\bullet$ is a connective complete Segal space object. The square

$$\begin{array}{ccc} (\Omega Y)_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y_0 \times Y_0 \end{array}$$

is definitionally a pullback square, so the counit map $\Omega Y_\bullet \to Y_\bullet$ always satisfies condition (2) of Definition 3.1.26. The induced map $* \simeq (\Omega Y)_0 \to |\mathrm{Gp}\, Y_\bullet|$ is an effective epimorphism by assumption, hence $\Omega Y_\bullet \to Y_\bullet$ is a Segal equivalence and the counit map $B\Omega Y_\bullet \to Y_\bullet$ an equivalence by Theorem 3.1.28, i.e. $Y_\bullet$ is contained in the essential image. □

3.2 The ∞-category of (∞, n)-categories

Following the previous Subsection, we regard $CSS_{\mathcal{Y}}$ as an ∞-category of $\mathcal{Y}$-enriched categories whenever $\mathcal{Y}$ is an absolute distributor. This allows us to inductively define the ∞-category of (∞, n)-categories in analogy to Proposition 2.3.10. There is a simplification insofar as the theory of ∞-categories is partially 'top-down' rather than 'bottom-up', i.e. we inductively construct the theory of (∞, r)-categories from the theory of spaces without any reference to (n, r)-categories for finite $n \geq r$.

Definition 3.2.1 ([Lur09a, Definition 1.3.6])**.** The ∞-category $\mathbf{Cat}_{(\infty,n)}$ of (∞, n)-categories is defined inductively by setting $\mathbf{Cat}_{(\infty,0)} \coloneqq \mathbf{Spc}$, the ∞-category of spaces, and $\mathbf{Cat}_{(\infty,n+1)} \coloneqq CSS_{\mathbf{Cat}_{(\infty,n)}}$ for

$n \geq 0$.

Remark 3.2.2. This is well-defined since **Spc** is an ∞-topos and by Proposition 3.1.30.

Remark 3.2.3. In the modern literature, the more prevalent approach is based on the notion of enrichment as in [GH15]. Their equivalence is established in [Hau15, Corollary 7.21].

The ∞-category $\mathbf{Cat}_{(\infty,1)}$ is not identical to the ∞-category $\mathbf{Cat}_\infty$, though an $(\infty,1)$-category is morally the same thing as a (weakly) **Spc**-enriched category. To justify this, the cosimplicial ∞-category $N(\Delta) \to \mathbf{Cat}_\infty$, $[n] \mapsto \Delta^n$ induces an adjunction ([Lur09b, Lemma 5.1.5.5, Corollary 5.2.6.5])

$$F\colon \operatorname{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Spc}\right) \rightleftarrows \mathbf{Cat}_\infty \colon G. \tag{4}$$

The left adjoint F is a left Kan extension and the right adjoint G is curried from the composite

$$N\left(\Delta\right)^{op} \times \mathbf{Cat}_\infty \longrightarrow \mathbf{Cat}_\infty^{op} \times \mathbf{Cat}_\infty \xrightarrow{\text{map}} \mathbf{Spc}.$$

Taking the explicit model $\operatorname{map}_{\mathbf{Cat}_\infty}(-,-) \simeq \operatorname{Fun}(-,-)^{\simeq}$, an ∞-category $\mathcal{C}$ gives rise to the simplicial space $G(\mathcal{C})\colon N\left(\Delta\right)^{op} \to \mathbf{Spc}$, $[n] \mapsto \operatorname{Fun}(\Delta^n, \mathcal{C})^{\simeq}$. For $n \geq 2$ and $0 \leq i \leq n$, the square

$$\begin{array}{ccc} \Delta^{\{i\}} & \hookrightarrow & \Delta^{\{i,\dots,n\}} \\ \downarrow & & \downarrow \\ \Delta^{\{0,\dots,i\}} & \hookrightarrow & \Delta^n \end{array}$$

is a homotopy pushout with respect to the Joyal model structure since $\Delta^{\{0,\dots,i\}} \cup \Delta^{\{i,\dots,n\}} \hookrightarrow \Delta^n$ is inner-anodyne. Applying $\operatorname{Fun}(-,\mathcal{C})$, we obtain a homotopy pullback square of ∞-categories, i.e. a pullback square in $\mathbf{Cat}_\infty$, which is preserved by the right adjoint $(-)^{\simeq}$. Thus, $G(\mathcal{C})$ is a category object in **Spc**.

Furthermore, the homotopy category of $G(\mathcal{C})$ as defined in [Lur09a, Definition 1.1.6] can be identified with the usual homotopy category $h\mathcal{C}$, so its invertible morphisms are precisely the equivalences in $\mathcal{C}$ and the groupoid object $G(\mathcal{C})^{\sim}$ as defined in [Lur09a, Notation 1.1.9] is given by the subsimplicial object $[n] \mapsto \operatorname{Fun}(\Delta^n, \mathcal{C}^{\simeq})$, which is constant as $\mathcal{C}^{\simeq}$ is a Kan complex and the standard simplices are weakly contractible. Thus, G factors through the full subcategory $CSS_{\mathbf{Spc}}$.

Theorem 3.2.4 ([Lur09a, Corollary 4.3.16])**.** *The adjunction* (4) *restricts to an adjoint equivalence*

$$F\colon CSS_{\mathbf{Spc}} \rightleftarrows \mathbf{Cat}_\infty \colon G. \qquad \square$$

Remark 3.2.5. This result is originally due to Joyal and Tierney.

Remark 3.2.6. Inductively, $\mathbf{Cat}_{(\infty,n)}$ is a strongly reflective subcategory of $\mathrm{Fun}\left(N\left(\Delta^{\times n}\right)^{op}, \mathbf{Spc}\right)$, consisting of the n-fold simplicial objects $X_{\bullet,\dots,\bullet}$ in $\mathbf{Spc}$ that satisfy:

- (Segal condition) For any $1 \leq i \leq n$ and $k_1, \dots, k_{i-1}, k_{i+1} \dots, k_n \geq 0$, the simplicial object $X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$ is a category object in $\mathbf{Spc}$.
- (Globularity) For any $1 \leq i \leq n$, the $(n-i)$-fold simplicial object $X_{1,\dots,1,0,\bullet,\dots,\bullet}$ in $\mathbf{Spc}$ is constant.
- (Completeness) For any $1 \leq i \leq n$, the simplicial object $X_{1,\dots,1,\bullet,0,\dots,0}$ (the number of ones is $i-1$) in $\mathbf{Spc}$ is complete.

Intuitively, the space $X_{1,\dots,1,0,\dots,0}$ (the number of ones is k) should be considered as the space of k-morphisms in this (∞,n)-category. The 'source' and 'target' $(k-1)$-morphism are given by the face maps $X_{1,\dots,1,1,0,\dots,0} \rightrightarrows X_{1,\dots,1,0,0,\dots,0}$, whereas the face maps on lower indices factor through degeneracy maps by globularity. In other words, we can also picture this as the space of 'k-cube-shaped' diagrams in $X_{\bullet,\dots,\bullet}$ whose faces are partially collapsed up to homotopy, according to the globularity conditions.

The Segal condition implies that a category object $X_\bullet$ in $\mathbf{Spc}$ is constant if and only the degeneracy map $s_0 \colon X_0 \to X_1$ is an equivalence. Thus, by Remark 3.1.16, it is complete if and only the map $X_0 \to X(K)$ is an equivalence. This in mind, let $[k_1, \dots, k_n]$ denote the image of $([k_1], \dots, [k_n])$ under the Yoneda embedding

$$N\left(\Delta^{\times n}\right) \hookrightarrow \mathrm{Fun}\left(N\left(\Delta^{\times n}\right)^{op}, \mathbf{Spc}\right) = \mathcal{P}\left(N\left(\Delta^{\times n}\right)\right).$$

The 'box product' $\boxtimes \colon \mathcal{P}\left(N\left(\Delta^{\times n}\right)\right) \times \mathcal{P}\left(N(\Delta^{\times m})\right) \to \mathcal{P}\left(N(\Delta^{\times(n+m)})\right)$ is the unique functor taking $([k_1, \dots, k_n], [k'_1, \dots, k'_m]) \mapsto [k_1, \dots, k_n, k'_1, \dots, k'_m]$ and preserving colimits in each variable separately. Then, $\mathbf{Cat}_{(\infty,n)}$ is the localization of $\mathcal{P}\left(N\left(\Delta^{\times n}\right)\right)$ at the maps of the form

$$\begin{aligned}
\mathrm{id}_{[k_1,\dots,k_{i-1}]} \boxtimes \left([1] \sqcup_{[0]} \dots \sqcup_{[0]} [1] \to [k_i]\right) \boxtimes \mathrm{id}_{[k_{i+1},\dots,k_n]}, &\qquad 1 \leq i \leq n,\ k_1, \dots, k_n \geq 0,\\
\mathrm{id}_{[1,\dots,1,0]} \boxtimes \left([0, \dots, 1, \dots, 0] \to [0, \dots, 0, \dots, 0]\right), &\qquad 1 \leq i \leq n,\ 1 \leq j \leq n-i,\\
\mathrm{id}_{[1,\dots,1]} \boxtimes \left([0] \sqcup_{[1]} [3] \sqcup_{[1]} [0] \to [0]\right) \boxtimes \mathrm{id}_{[0,\dots,0]}, &\qquad 1 \leq i \leq n.
\end{aligned}$$

The local objects with respect to the first two classes, i.e. which satisfy the Segal and globularity conditions, are the *n-fold Segal spaces*. The ones additionally local with respect to the third class, i.e. which also satisfy the completeness condition, are the *n-fold complete Segal spaces*. These were first introduced as a model for (∞,n)-categories in [Bar05].

Now, we establish the expected properties of the ∞-categories $\mathbf{Cat}_{(\infty,n)}$, $n \geq 0$.

Proposition 3.2.7. *The ∞-category $\mathbf{Cat}_{(\infty,n)}$ is presentable, $n \geq 0$.* □

First, we want to construct the adjoint triple analogous to that of Proposition 2.3.12.

Proposition 3.2.8. *The inductive application of Proposition 3.1.36 to the adjoint triple*

$$\mathbf{Spc} \underset{\mathrm{ev}_0}{\overset{|-|}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} CSS_{\mathbf{Spc}} \quad (\delta)$$

yields adjoint triples

$$\mathbf{Cat}_{(\infty,n)} \underset{\kappa_n}{\overset{\tau_n}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} \mathbf{Cat}_{(\infty,n+1)} \ , \qquad \forall n \geq 0. \quad (\varepsilon_n)$$

Proof. The adjoint triple restricts to an adjoint triple of equivalences between **Spc** and $\mathbf{Spc}'$. The adjunction $|-| \dashv \delta$ satisfies the hypothesis of Proposition 3.1.36, (1) by Lemma 3.1.32 and this implies that the adjunction $\delta \dashv \mathrm{ev}_0$ satisfies the hypothesis of Proposition 3.1.36, (2). The Proposition then follows by induction (and full faithfulness by Proposition 3.1.39). □

Remark 3.2.9. In terms of simplicial sets, there are explicit formulae

$$(\varepsilon_n X)_{k_1,\dots,k_{n+1}} = X_{k_1,\dots,k_n}, \qquad (\kappa_n X)_{k_1,\dots,k_n} = X_{k_1,\dots,k_n,0}.$$

The composite adjoint triple between $\mathbf{Cat}_{(\infty,n+k)}$ and $\mathbf{Cat}_{(\infty,n)}$ will be denoted as $\tau_n^k \dashv \varepsilon_n^k \dashv \kappa_n^k$ for any $n, k \geq 0$. Furthermore, we denote $|-| \coloneqq \tau_0^n$ and $(-)^{\simeq} \coloneqq \kappa_0^n$ for any $n \geq 0$.

Now, we want to prove the analogue of Theorem 2.2.7.

Proposition 3.2.10. *The ∞-category $\mathbf{Cat}_{(\infty,n)}$ is cartesian closed, $n \geq 0$.*

Proof. This is an immediate induction using Theorem 3.1.41, starting with the absolute distributor **Spc**, which is cartesian closed by Example A.0.4 or A.0.5. □

Proposition 3.2.11. *For any $n \geq 0$, the functor $\tau_n \colon \mathbf{Cat}_{(\infty,n+1)} \to \mathbf{Cat}_{(\infty,n)}$ is left exact relative to the essential image of $\varepsilon_0^{n+1} \colon \mathbf{Spc} \hookrightarrow \mathbf{Cat}_{(\infty,n+1)}$. In particular, it preserves finite products.*

Proof. This follows from the inductive application of Proposition 3.1.36, (1). □

Corollary 3.2.12. *For any $n \geq 0$, the essential image of $\varepsilon_n \colon \mathbf{Cat}_{(\infty,n)} \hookrightarrow \mathbf{Cat}_{(\infty,n+1)}$ is an exponential ideal.*

Proof. This is immediate by Proposition 3.2.11 and Proposition A.0.9. □

Proposition 3.2.13. *In the ∞-category $\mathbf{Cat}_{(\infty,n)}$, $n \geq 0$, small filtered colimits commute with finite limits.*

Proof. This is an immediate induction using [Lur09a, Proposition 1.2.29], the hypotheses of the base case being satisfied by [Lur09b, Proposition 5.3.3.3]. □

Remark 3.2.14. In particular, filtered colimits are computed pointwise.

Proposition 3.2.15. *For any $n \geq 0$, the functor $\kappa_n \colon \mathbf{Cat}_{(\infty,n+1)} \to \mathbf{Cat}_{(\infty,n)}$ preserves small filtered colimits.*

Proof. The proof of Proposition 3.2.13 implies that $\mathbf{Cat}_{(\infty,n+1)}$ is closed under small filtered colimits in $\mathrm{Fun}\left(N\left(\Delta\right)^{op}, \mathbf{Cat}_{(\infty,n)}\right)$ for any $n \geq 0$. The claim then follows by induction since colimits in functor categories can be computed pointwise. □

Proposition 3.2.16. *The ∞-category $\mathbf{Cat}_{(\infty,n)}$ is compactly generated, $n \geq 0$.*

Proof. The strongly reflective subcategory $\mathbf{Cat}_{(\infty,n)}$ of $\mathcal{P}\left(N\left(\Delta^{\times n}\right)\right)$ is stable under small filtered colimits by the proof of Proposition 3.2.13. The ∞-category $\mathcal{P}\left(N\left(\Delta^{\times n}\right)\right)$ is ω-accessible by [Lur09b, Proposition 5.3.5.12], so [Lur09b, Corollary 5.5.7.3] implies the result. □

Remark 3.2.17. This entire section holds true replacing $\mathbf{Spc}$ with an arbitrary ∞-topos $\mathcal{X}$. This yields the '∞-categories of stacks of (∞, n)-categories on $\mathcal{X}$'.

3.3 The ∞-category of (∞, ∞)-categories

In analogy to Proposition 2.3.11, we define the ∞-category of (∞, ∞)-categories. Instead of the ad hoc constructions of Subsection 2.3, we will employ a general principle for constructing adjunctions between sequential limits that is explained in Appendix B.

Definition 3.3.1. The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ of (small) (∞, ∞)-categories is the limit (in $\mathbf{CAT}_\infty$) of

$$\ldots \xrightarrow{\kappa_3} \mathbf{Cat}_{(\infty,3)} \xrightarrow{\kappa_2} \mathbf{Cat}_{(\infty,2)} \xrightarrow{\kappa_1} \mathbf{Cat}_{(\infty,1)} \xrightarrow{\kappa_0} \mathbf{Spc}.$$

In particular, an (∞, ∞)-*category* is a sequence $(\mathcal{C}_n)_{n\geq 0}$, where $\mathcal{C}_n$ is an (∞, n)-category and $\kappa_n(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$.

Remark 3.3.2. An immediate induction using Proposition 3.1.38 implies that the κ_n, $n \geq 0$ are isofibrations, hence this ∞-categorical limit is indeed modeled by the strict limit of simplicial sets.

Remark 3.3.3. This corresponds to the so-called *inductive* definition of (∞, ∞)-categories. See [HL25, Section 1.2] for a discussion on the ambiguity in defining (∞, ∞)-categories.

Proposition 3.3.4. *The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ is compactly generated.*

Proof. This follows from [Lur09b, Proposition 5.5.7.6] since the ∞-categories $\mathbf{Cat}_{(\infty,n)}$, $n \geq 0$ are compactly generated by Proposition 3.2.16 and the functors κ_n, $n \geq 0$ preserve small limits as they are right adjoints as well as small filtered colimits by Proposition 3.2.15. □

Remark 3.3.5. In particular, letting $\Delta^{\times\infty} \coloneqq \operatorname{colim}_{n\geq 0} \Delta^{\times n} = \bigcup_{n\geq 0} \Delta^{\times n}$ denote the 'infinite polysimplex category' (here, $\Delta^{\times n} \hookrightarrow \Delta^{\times(n+1)}$, $([k_1], \dots, [k_n]) \mapsto ([k_1], \dots, [k_n], [0])$), we can consider $\mathbf{Cat}_{(\infty,\infty)}$ as a full subcategory of $\operatorname{Fun}\left(N\left(\Delta^{\times\infty}\right)^{op}, \mathbf{Spc}\right)$. Following the discussion in Remark 3.2.6, we see that a functor $X\colon N(\Delta^{\times\infty}) \to \mathbf{Spc}$ is an (∞,∞)-category if and only if it satisfies the Segal condition in each variable separately, the poly-simplicial object $X_{1,\dots,1,0,\bullet,\dots}$ is constant and the simplicial object $X_{1,\dots,1,\bullet,0,\dots}$ is complete for any number of ones. Thus, by [Lur09b, Propositions 5.5.1.2 and 5.5.4.15], we can regard $\mathbf{Cat}_{(\infty,\infty)}$ as an accessible localization of $\operatorname{Fun}\left(N\left(\Delta^{\times\infty}\right)^{op}, \mathbf{Spc}\right)$. This could also be deduced from Theorem B.0.4, using Corollary 3.3.9.

There is an analogue of Proposition 2.3.3.

Proposition 3.3.6. *The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ is an absolute distributor and there is a canonical equivalence $CSS_{\mathbf{Cat}_{(\infty,\infty)}} \simeq \mathbf{Cat}_{(\infty,\infty)}$.*

Proof. The first claim is a consequence of [Gol24, Proposition 3.2.3]. There is a canonical isomorphism $\Delta \times \Delta^{\times\infty} \cong \Delta^{\times\infty}$, which induces an equivalence

$$\operatorname{Fun}\left(N\left(\Delta\right)^{op}, \operatorname{Fun}\left(N\left(\Delta^{\times\infty}\right)^{op}, \mathbf{Spc}\right)\right) \simeq \operatorname{Fun}\left(N\left(\Delta^{\times\infty}\right)^{op}, \mathbf{Spc}\right).$$

This equivalence restricts, by Remark 3.3.5, to the desired equivalence $CSS_{\mathbf{Cat}_{(\infty,\infty)}} \simeq \mathbf{Cat}_{(\infty,\infty)}$. □

Lemma 3.3.7. *Let K be a small simplicial set and $\mathbf{CAT}_\infty^K$ the (non-full) subcategory of $\mathbf{CAT}_\infty$ consisting of those (not necessarily small) ∞-categories that admit K-indexed colimits and functors that preserve K-indexed colimits. This subcategory is stable under small limits and, given a small limit diagram $\overline{p}\colon S^{\triangleleft} \to \mathbf{CAT}_\infty^K \subseteq \mathbf{CAT}_\infty$, the functors $\overline{p}(\infty) \to \overline{p}(x)$ for all vertices $x \in S$ jointly reflect K-indexed colimits. Furthermore, they are jointly conservative.*

Remark 3.3.8. The same conclusion applies to the op-dual subcategory consisting of those (not necessarily small) ∞-categories that admit K-indexed limits and functors that preserve them.

Proof. The techniques of [Lur09b, Proposition 4.4.2.6] reduce this to verifying the Lemma for small products and pullbacks. The former is a special case of [Lur09b, Corollary 5.1.2.3]. The latter is [Lur09b, Lemma 5.4.5.5]. The last claim is evident in either case. □

Corollary 3.3.9. *The canonical projections* $\kappa_{\leq n}\colon \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,n)}$, $n \geq 0$ *preserve and jointly reflect small limits as well as small filtered colimits.*

Proof. The functors κ_n, $n \geq 0$ preserve small limits as well as small filtered colimits by Proposition 3.2.15, so the claim follows by Lemma 3.3.7. □

Remark 3.3.10. In particular, small limits as well as small filtered colimits of (∞,∞)-categories can be computed levelwise. A method of constructing arbitrary small colimits can be extracted from Theorem B.0.4, though we have not found an application for that.

Proposition 3.3.11. *In the* ∞*-category* $\mathbf{Cat}_{(\infty,\infty)}$*, small filtered colimits commute with finite limits.*

Proof. Let J be a small filtered simplicial set, K a finite simplicial set and consider a limit diagram $\overline{p}\colon K^{\triangleleft} \to \mathrm{Fun}\left(J, \mathbf{Cat}_{(\infty,\infty)}\right)$. For $n \geq 0$, consider the diagram

$$\begin{array}{ccccc} K^{\triangleleft} & \xrightarrow{\overline{p}} & \mathrm{Fun}\left(J, \mathbf{Cat}_{(\infty,\infty)}\right) & \xrightarrow{\mathrm{colim}_J} & \mathbf{Cat}_{(\infty,\infty)} \\ & & \downarrow{\scriptstyle (\kappa_{\leq n})_*} & & \downarrow{\scriptstyle \kappa_{\leq n}} \\ & & \mathrm{Fun}\left(J, \mathbf{Cat}_{(\infty,n)}\right) & \xrightarrow{\mathrm{colim}_J} & \mathbf{Cat}_{(\infty,n)}. \end{array}$$

The functor $\kappa_{\leq n}$ preserves J-indexed colimits by Corollary 3.3.9, so the square is commutative. The functor $(\kappa_{\leq n})_*$ preserves (finite) limits since $\kappa_{\leq n}$ does and J-indexed colimits in $\mathbf{Cat}_{(\infty,n)}$ are left exact by Proposition 3.2.13, hence $\kappa_{\leq n} \circ \mathrm{colim}_J \circ \overline{p} \simeq \mathrm{colim}_J \circ (\kappa_{\leq n})_* \circ \overline{p}$ is a limit diagram. Thus, by Corollary 3.3.9, $\mathrm{colim}_J \circ \overline{p}$ is a limit diagram, as desired. □

Proposition 3.3.12. *For any* $n \geq 0$*, the fully faithful functors* $\varepsilon_n^k\colon \mathbf{Cat}_{(\infty,n)} \hookrightarrow \mathbf{Cat}_{(\infty,n+k)}$, $k \geq 0$ *induce a fully faithful functor* $\varepsilon_{\leq n}\colon \mathbf{Cat}_{(\infty,n)} \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$*, left adjoint to* $\kappa_{\leq n}\colon \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,n)}$.

Proof. This is an application of Corollary B.0.6. □

Proposition 3.3.13. *The counits of the adjunctions* $\varepsilon_{\leq n} \dashv \kappa_{\leq n}$ *induce a natural equivalence*

$$\operatorname*{colim}_{n\geq 0} \varepsilon_{\leq n}\kappa_{\leq n} \xrightarrow{\sim} \mathrm{id}_{\mathbf{Cat}_{(\infty,\infty)}}.$$

Proof. This is an application of Corollary B.0.8. □

Proposition 3.3.14. *For any* $n \geq 0$*, the fully faithful functor* $\varepsilon_{\leq n}\colon \mathbf{Cat}_{(\infty,n)} \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$ *admits a left adjoint* $\tau_{\leq n}\colon \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,n)}$*, given by* $\mathcal{C} = (\mathcal{C}_n)_{n\geq 0} \mapsto \mathrm{colim}_{k\geq 0}\, \tau_n^k \mathcal{C}_{n+k}$.

Proof. The diagram

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\mathrm{id}} & \mathbf{Cat}_{(\infty,n)} & \xrightarrow{\mathrm{id}} & \mathbf{Cat}_{(\infty,n)} & \xrightarrow{\mathrm{id}} & \dots & \xrightarrow{\mathrm{id}} & \mathbf{Cat}_{(\infty,n)} \\
 & & \downarrow{\scriptstyle \varepsilon_n^{m+1}} & & \downarrow{\scriptstyle \varepsilon_n^{m}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
\dots & \xrightarrow{\kappa_{n+m+1}} & \mathbf{Cat}_{(\infty,n+m+1)} & \xrightarrow{\kappa_{n+m}} & \mathbf{Cat}_{(\infty,n+m)} & \xrightarrow{\kappa_{n+m-1}} & \dots & \xrightarrow{\kappa_n} & \mathbf{Cat}_{(\infty,n)}.
\end{array}$$

admits an application of Theorem B.0.4. □

Proposition 3.3.15. *The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ is cartesian closed.*

Proof. For any (∞,∞)-category $\mathcal{C} = (\mathcal{C}_n)_{n\geq 0}$, the diagram

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\kappa_{n+1}} & \mathbf{Cat}_{(\infty,n+1)} & \xrightarrow{\kappa_n} & \mathbf{Cat}_{(\infty,n)} & \xrightarrow{\kappa_{n-1}} & \dots & \xrightarrow{\kappa_0} & \mathbf{Cat}_{(\infty,0)} \\
 & & \downarrow{\scriptstyle \mathcal{C}_{n+1}\times(-)} & & \downarrow{\scriptstyle \mathcal{C}_{n}\times(-)} & & & & \downarrow{\scriptstyle \mathcal{C}_{0}\times(-)} \\
\dots & \xrightarrow{\kappa_{n+1}} & \mathbf{Cat}_{(\infty,n+1)} & \xrightarrow{\kappa_n} & \mathbf{Cat}_{(\infty,n)} & \xrightarrow{\kappa_{n-1}} & \dots & \xrightarrow{\kappa_0} & \mathbf{Cat}_{(\infty,0)}.
\end{array}$$

admits an application of (the dual of) Theorem B.0.4. Thus, $\mathcal{C}\times(-)$ admits a left adjoint. □

Remark 3.3.16. An alternative argument using the adjoint functor theorem can be found in [Ste21, Proposition 3.6.10].

Proposition 3.3.17. *For any $n \geq 0$, the functor $\tau_{\leq n}\colon \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,n)}$ is left exact relative to the essential image of $\varepsilon_{\leq 0}\colon \mathbf{Spc} \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$. In particular, it preserves finite products.*

Proof. The composite $\tau_{\leq n}\varepsilon_{\leq 0} \simeq \varepsilon_0^n$ since it is left adjoint to $\kappa_{\leq 0}\varepsilon_{\leq n} \simeq \kappa_0^n$. Thus, it remains to demonstrate that $\tau_{\leq n}$ preserves the pullback of any cospan $\mathcal{C} \to X \leftarrow \mathcal{D}$ of (∞,∞)-categories, where X is contained in the essential image of $\varepsilon_{\leq 0}$. There is a commutative square

$$\begin{array}{ccc}
\tau_{\leq n}(\mathcal{C}\times_X \mathcal{D}) & \longrightarrow & \tau_{\leq n}(\mathcal{C}) \times_X \tau_{\leq n}(\mathcal{D}) \\
\wr\uparrow & & \wr\uparrow \\
\mathrm{colim}_{k\geq 0}\, \tau_n^k\left(\kappa_{\leq n+k}\mathcal{C}\times_X \kappa_{\leq n+k}\mathcal{D}\right) & \xrightarrow{\sim} & \mathrm{colim}_{k\geq 0}\left(\tau_n^k(\kappa_{\leq n+k}\mathcal{C})\times_X \tau_n^k(\kappa_{\leq n+k}\mathcal{D})\right).
\end{array}$$

The vertical maps are equivalences by Proposition 3.3.14, Corollary 3.3.9 and Proposition 3.2.13. The bottom horizontal map is an equivalence since τ_n^k preserves pullbacks over the essential image of ε_0^{n+k} for any $k \geq 0$ by Proposition 3.2.11. Thus, the upper horizontal map is an equivalence. □

Corollary 3.3.18. *For any $n \geq 0$, the essential image of $\varepsilon_{\leq n}\colon \mathbf{Cat}_{(\infty,n)} \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$ is an exponential ideal.*

Proof. This is immediate by Proposition 3.3.17 and Proposition A.0.9. □

3.4 Towards Spectra

In this Subsection, we define the ∞-category of categorical spectra, guided by the table of analogies of Subsection 1.2. Though it is not evident that our formalism is equivalent to theirs, this Subsection is primarily based on [Ste21, Chapter 13] and [Mas24, Chapter 3]. The reader is assumed to be familiar with the theory of (cartesian) symmetric monoidal ∞-categories and algebras over the little cube operads $\mathbb{E}_k$ for $k = 0, 1, \ldots, \infty$, see [Lur17, Chapters 2.4 and 5.1].

Definition 3.4.1. The *∞-category of symmetric monoidal (∞,∞)-categories* is

$$\mathbf{Cat}^{\otimes}_{(\infty,\infty)} \coloneqq \mathrm{CAlg}\left(\mathbf{Cat}_{(\infty,\infty)}\right) = \mathrm{Alg}_{\mathbb{E}_\infty}\left(\mathbf{Cat}_{(\infty,\infty)}\right).$$

Remark 3.4.2. Here, $\mathbf{Cat}_{(\infty,\infty)}$ is regarded as a cartesian symmetric monoidal ∞-category. By [Lur17, Proposition 2.4.2.5], a symmetric monoidal (∞,∞)-category is equivalently a functor $F\colon N(\mathcal{F}\mathrm{in}_*) \to \mathbf{Cat}_{(\infty,\infty)}$, such that the images of the maps $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ under F (see [Lur17, Notation 2.0.0.2] for the notation) exhibit $F(\langle n\rangle)$ as an n-fold product of $F(\langle 1\rangle)$, for all $n \geq 0$. Following [Lur17, Remark 2.4.2.3], we interpret F as giving the (∞,∞)-category $F(\langle 1\rangle)$ together with a multiplication law that is unital, associative and commutative up to coherent homotopy.

Proposition 3.4.3. *(1) The ∞-category $\mathbf{Cat}^{\otimes}_{(\infty,\infty)}$ is presentable.*

(2) The forgetful functor $\mathbf{Cat}^{\otimes}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,\infty)}$ preserves and reflects small limits as well as small sifted colimits.

(3) In the ∞-category $\mathbf{Cat}^{\otimes}_{(\infty,\infty)}$, small filtered colimits commute with finite limits.

(4) The ∞-category $\mathbf{Cat}^{\otimes}_{(\infty,\infty)}$ admits finite biproducts.

Proof. The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ is presentable by Proposition 3.3.4 and the cartesian product

$$(-)\times(-) : \mathbf{Cat}_{(\infty,\infty)} \times \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,\infty)}$$

preserves colimits in each variable separately by Proposition 3.3.15, hence $\mathbf{Cat}^{\otimes}_{(\infty,\infty)}$ is presentable by [Lur17, Corollary 3.2.3.5]. The characterization of small limits resp. small sifted colimits is [Lur17, Corollary 3.2.2.4] resp. [Lur17, Corollary 3.2.3.2]. This immediately implies (3) by Proposition 3.3.11 and [Lur09b, Example 5.5.8.3]. Lastly, (4) holds by [Lur17, Proposition 3.2.4.7]. □

Now, we specialize the delooping hypothesis to (∞,∞)-categories.

Warning. In [Lur17, Remark 1.1.2.6], $\Omega_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ denotes the functor $X \mapsto \lim(* \to X \leftarrow *)$ for any pointed ∞-category $\mathcal{C}$. The functors Ω appearing in this Subsection are *not* of this form (instead, they

can be thought of as lax limits of such cospans). This is consistent with the notation of [Ste21] and [Mas24]. Further justification is provided by Proposition 3.4.13.

Proposition 3.4.4. *There is a canonical adjunction*

$$B\colon \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \rightleftarrows \mathbf{Cat}_{(\infty,\infty),*}\colon \Omega.$$

The left adjoint B is fully faithful, preserves finite products and its essential image consists of the connective (pointed) (∞,∞)-categories, i.e. those for which $\pi_0(\mathcal{C}^{\simeq})$ is a singleton.

Proof. This is the adjunction of Theorem 3.1.44 for the absolute distributor $\mathbf{Cat}_{(\infty,\infty)}$ of Proposition 3.3.6, combined with the equivalence $\mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \xrightarrow{\sim} \mathrm{Mon}\left(\mathbf{Cat}_{(\infty,\infty)}\right)$ of [Lur17, Propositions 2.4.2.5 and 4.1.2.10, Example 5.1.0.7], the equivalence $CSS_{\mathbf{Cat}_{(\infty,\infty)}} \simeq \mathbf{Cat}_{(\infty,\infty)}$ of Proposition 3.3.6 and the characterization of effective epimorphisms in $\mathbf{Spc}$ of [Lur09b, Corollary 7.2.1.15]. □

Abusing notation, the composite $\mathbf{Cat}_{(\infty,\infty),*} \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \to \mathbf{Cat}_{(\infty,\infty),*}$ will also be denoted $\Omega\colon \mathbf{Cat}_{(\infty,\infty),*} \to \mathbf{Cat}_{(\infty,\infty),*}$.

Corollary 3.4.5. *For any $n \geq 0$, there is a canonical adjunction*

$$B^n\colon \mathrm{Alg}_{\mathbb{E}_n}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \rightleftarrows \mathbf{Cat}_{(\infty,\infty),*}\colon \Omega^n.$$

The left adjoint B^n is fully faithful, preserves finite products and its essential image consists of the n-connective (pointed) (∞,∞)-categories, i.e. those for which $\Omega^k\mathcal{C}$ is connective for all $k = 0,\dots,n-1$. Here, the composite $\mathbf{Cat}_{(\infty,\infty),*} \to \mathrm{Alg}_{\mathbb{E}_n}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \to \mathbf{Cat}_{(\infty,\infty),*}$ of Ω^n and the forgetful functor is the n-fold composite of the endofunctor $\Omega\colon \mathbf{Cat}_{(\infty,\infty),*} \to \mathbf{Cat}_{(\infty,\infty),*}$.

Proof. Any adjunction $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ of finite product-preserving functors between ∞-categories admitting finite products induces an adjunction $L_*\colon \mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{C}\right) \rightleftarrows \mathrm{Fun}\left(N\left(\Delta\right)^{op},\mathcal{D}\right) : R_*$, which restricts to an adjunction $\mathrm{Alg}_{\mathbb{E}_1}\left(\mathcal{C}\right) \simeq \mathrm{Mon}(\mathcal{C}) \rightleftarrows \mathrm{Mon}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathbb{E}_1}\left(\mathcal{D}\right)$. The properties of preserving finite products and being fully faithful are stable under this construction. The claim now follows by induction. The case $n = 0$ is the canonical equivalence $\mathrm{Alg}_{\mathbb{E}_0}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \simeq \mathbf{Cat}_{(\infty,\infty),*}$ of [Lur17, Remark 2.1.3.10]. In the inductive step, we construct the composite adjunction

$$\begin{aligned}
\mathrm{Alg}_{\mathbb{E}_{n+1}}\left(\mathbf{Cat}_{(\infty,\infty)}\right) &\simeq \mathrm{Alg}_{\mathbb{E}_1}\left(\mathrm{Alg}_{\mathbb{E}_n}\left(\mathbf{Cat}_{(\infty,\infty)}\right)\right)\\
&\rightleftarrows \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty),*}\right)\\
&\simeq \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right)\\
&\rightleftarrows \mathbf{Cat}_{(\infty,\infty),*},
\end{aligned}$$

where the equivalences are due to the base case and the Dunn Additivity Theorem ([Lur17, Theorem 5.1.2.2]), and we have used Proposition 3.4.4. The essential image is also characterized inductively, the base case being trivial. In one direction, note that, for any $\mathbb{E}_n$-monoidal (∞,∞)-category $\mathcal{C}$ and $k=0,\dots,n-1$, $\Omega^k B^n\mathcal{C} \simeq B^{n-k}\mathcal{C}$ (even as $\mathbb{E}_k$-monoidal (∞,∞)-categories) is contained in the essential image of B, hence connective. In the converse direction, suppose that $\mathcal{C}$ is a pointed (∞,∞)-category and $\Omega^k\mathcal{C}$ is connective for $k=0,\dots,n-1$. The counit map of this adjunction is the composite

$$B^n\Omega^n\mathcal{C} = B^{n-1}B\Omega\Omega^{n-1}\mathcal{C} \to B^{n-1}\Omega^{n-1}\mathcal{C} \xrightarrow{\sim} \mathcal{C},$$

where the second map is an equivalence by the induction hypothesis. Since $\Omega^{n-1}\mathcal{C}$ is connective by assumption, the counit map $B\Omega\Omega^{n-1}\mathcal{C} \to \Omega^{n-1}\mathcal{C}$ is an equivalence of pointed (∞,∞)-categories. The forgetful functor $\mathrm{Alg}_{\mathbb{E}_{n-1}}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \to \mathbf{Cat}_{(\infty,\infty),*}$ is conservative ([Lur17, Remark 2.1.3.8]), so it is even an equivalence of of $\mathbb{E}_{n-1}$-monoidal (∞,∞)-categories. Thus, its image under B^{n-1} and hence the couint map are equivalences of pointed (∞,∞)-categories, as desired. □

Remark 3.4.6. For any $k \geq 0$, this restricts to $B^n\colon \mathrm{Alg}_{\mathbb{E}_n}\left(\mathbf{Cat}_{(\infty,k)}\right) \rightleftarrows \mathbf{Cat}_{(\infty,n+k),*}\colon \Omega^n$.

Corollary 3.4.7. *For any $n \geq 0$, there is a canonical adjunction*

$$B^n\colon \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \rightleftarrows \mathbf{Cat}^{\otimes}_{(\infty,\infty)}\colon \Omega^n.$$

The left adjoint B^n is fully faithful, preserves finite products and its essential image consists of the n-connective (symmetric monoidal) (∞,∞)-categories.

Proof. This is an analogous argument to that of Corollary 3.4.5, directly applying $\mathrm{Alg}_{\mathbb{E}_\infty}(-)$. In this case, B^n resp. Ω^n is the literal n-fold composite of B resp. Ω. □

Definition 3.4.8. The ∞-category **CatSp** of *categorical spectra* is the limit (in $\mathbf{CAT}_\infty$) of

$$\dots \xrightarrow{\Omega} \mathbf{Cat}_{(\infty,\infty),*} \xrightarrow{\Omega} \mathbf{Cat}_{(\infty,\infty),*} \xrightarrow{\Omega} \mathbf{Cat}_{(\infty,\infty),*}.$$

In particular, a *categorical spectrum* consists of a sequence $(\mathcal{C}_n)_{n\geq 0}$ of pointed (∞,∞)-categories together with specified equivalences $\mathcal{C}_n \xrightarrow{\sim} \Omega\mathcal{C}_{n+1}$ of pointed (∞,∞)-categories for all $n \geq 0$. The projection onto the n-th term is denoted $\Omega^{\infty-n}\colon \mathbf{CatSp} \to \mathbf{Cat}_{(\infty,\infty),*}$.

Remark 3.4.9. Similar to how we can think of spectra as generalized spaces that admit negative-dimensional homotopy groups, a categorical spectrum can be thought of as a (pointed) higher category-like structure that admits 'negative-dimensional cells', see [Ker24, Theorem 3.1.7] for a precise interpretation of this idea, or the discussion surrounding [Ste21, Definition 13.2.12].

Proposition 3.4.10. *There is a canonical adjunction*

$$\Sigma_{\mathrm{cat}} \coloneqq B \circ \mathrm{Free}_{\mathbb{E}_1/\mathbb{E}_0} \colon \mathbf{Cat}_{(\infty,\infty),*} \rightleftarrows \mathbf{Cat}_{(\infty,\infty),*} \colon \Omega.$$

In particular, Ω preserves small limits as well as small filtered colimits.

Proof. This is obtained by composing the adjunctions $B\colon \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \rightleftarrows \mathbf{Cat}_{(\infty,\infty),*} \colon \Omega$ of Proposition 3.4.4 and $\mathrm{Free}_{\mathbb{E}_1/\mathbb{E}_0}\colon \mathbf{Cat}_{(\infty,\infty),*} \simeq \mathrm{Alg}_{\mathbb{E}_0}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \rightleftarrows \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right)$ of [Lur17, Corollary 3.1.3.5], the hypotheses of which are satisfied by Proposition 3.3.15. □

Proposition 3.4.11. *(1) The category* **CatSp** *is compactly generated.*

(2) The forgetful functors $\Omega^{\infty-n}\colon \mathbf{CatSp} \to \mathbf{Cat}_{(\infty,\infty),*}$, $n \geq 0$ preserve and jointly reflect small limits as well as small filtered colimits.

(3) In the ∞-category **CatSp***, small filtered colimits commute with finite limits.*

(4) The induced functor $\Omega\colon \mathbf{CatSp} \to \mathbf{CatSp}$ is an equivalence.

Proof. The ∞-category $\mathbf{Cat}_{(\infty,\infty)}$ is compactly generated by Proposition 3.3.4, hence so is $\mathbf{Cat}_{(\infty,\infty),*}$ by [Lur09b, Proposition 5.5.3.11] (to get compact generation and not just presentability, choose $\tau = \kappa = \omega$ in [Lur09b, Corollary 5.4.5.16]). Together with Proposition 3.4.10, this implies that the diagram defining **CatSp** is contained in Pr^R_ω, so (1) follows from [Lur09b, Proposition 5.5.7.6]. Next, (2) follows from Lemma 3.3.7 and Proposition 3.4.10. This implies (3) by Proposition 3.3.11 (and [Lur09b, Propositions 1.2.13.8, 4.4.2.9]). Lastly, (4) is diagramatically immediate. □

Remark 3.4.12. Following the previous Warning, note that (4) does not mean that [Lur17, Corollary 1.4.2.27] applies. To the contrary, **CatSp** is *not* a stable ∞-category, though we won't prove this here.

Proposition 3.4.13. *The square*

$$\begin{array}{ccc} \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} \\ \uparrow & & \uparrow \\ \mathbf{Spc}_* & \xrightarrow{\Omega} & \mathbf{Spc}_* \end{array}$$

is commutative, where $\Omega\colon \mathbf{Spc}_* \to \mathbf{Spc}_*$ is the usual loop space functor.

Proof. The loop space of a pointed Kan complex X is computed as the homotopy pullback

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X. \end{array}$$

The loop object of X considered as a pointed (∞,∞)-category is computed as the homotopy pullback (in $\mathbf{Spc}$, since $\varepsilon_{\leq 0}\colon \mathbf{Spc} \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$ preserves small limits)

$$\begin{array}{ccc} \Omega X & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \times X. \end{array}$$

Factoring $* \xhookrightarrow{\sim} \mathrm{Map}(\Delta^1, X) \times_X \{*\} \twoheadrightarrow X$ resp. $X \xhookrightarrow{\sim} \mathrm{Map}(\Delta^1, X) \twoheadrightarrow X \times X$, we see that either is modeled by the strict pullback $\mathrm{Map}(\Delta^1, X) \times_{X\times X} \{*\}$. □

Remark 3.4.14. This square is vertically neither left nor right adjointable. Indeed, consider $B\mathbb{N}$ as a pointed (∞,∞)-category, then $\Omega B\mathbb{N} \simeq \mathbb{N}$ is already a (discrete) space. However, $\Omega(B\mathbb{N})^{\simeq} \simeq \Omega * \simeq *$ and $\Omega\,|B\mathbb{N}| \simeq \Omega B\mathbb{Z} \simeq \mathbb{Z}$ (and the natural map corresponds to the canonical injection $\mathbb{N} \hookrightarrow \mathbb{Z}$).

Corollary 3.4.15. *A pointed space X is k-connective as an (∞,∞)-category if and only if it is k-connected in the usual sense.* □

Corollary 3.4.16. *There is a canonical fully faithful functor $\mathbf{Sp} \hookrightarrow \mathbf{CatSp}$, which admits both adjoints. The left adjoint $|-| : \mathbf{CatSp} \to \mathbf{Sp}$, called geometric realization, is given for a categorical spectrum $\mathcal{C} = (\mathcal{C}_n)_n$ by*

$$|\mathcal{C}| \simeq \left(\operatorname*{colim}_{k\geq 0} \Omega^k\, |\mathcal{C}_{n+k}| \right)_{n\geq 0}.$$

It is left exact relative to the essential image of $\mathbf{Sp} \hookrightarrow \mathbf{CatSp}$. In particular, it preserves finite products.

Remark 3.4.17. In other words, we can apply the geometric realization $|-| : \mathbf{Cat}_{(\infty,\infty),*} \to \mathbf{Spc}_*$ levelwise to obtain the spaces $|\mathcal{C}_n|$, $n \geq 0$ and canonical maps $|\mathcal{C}_n| \simeq |\Omega\mathcal{C}_{n+1}| \to \Omega\,|\mathcal{C}_{n+1}|$. These turn $(|\mathcal{C}_n|)_n$ into a pre-spectrum and the geometric realization $|\mathcal{C}|$ is precisely its spectrification.

Proof. The inclusion $\mathbf{Spc}_* \hookrightarrow \mathbf{Cat}_{(\infty,\infty),*}$ has a left adjoint $|-|$ and right adjoint $(-)^{\simeq}$ by Propositions 3.3.12 and 3.3.14, and [Lur09b, Proposition 5.2.5.1]. The conclusion follows by applying Corollary B.0.2 as well as Theorem B.0.4 (and its dual) to the diagram

$$\begin{array}{ccccccc} \ldots & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} \\ & & \uparrow & & \uparrow & & \uparrow \\ \ldots & \xrightarrow{\Omega} & \mathbf{Spc}_* & \xrightarrow{\Omega} & \mathbf{Spc}_* & \xrightarrow{\Omega} & \mathbf{Spc}_*, \end{array}$$

given by Proposition 3.4.13, which satisfies the hypotheses by Proposition 3.4.10.

For a cospan $\mathcal{C} \to X \leftarrow \mathcal{D}$ of categorical spectra, where X is an ordinary spectrum, we calculate

$$\begin{aligned}
|\mathcal{C} \times_X \mathcal{D}| &\simeq \left(\operatorname*{colim}_{k \geq 0} \Omega^k \left| \mathcal{C}_{n+k} \times_{X_{n+k}} \mathcal{D}_{n+k} \right| \right)_{n \geq 0} \\
&\simeq \left(\operatorname*{colim}_{k \geq 0} \left(\Omega^k \left| \mathcal{C}_{n+k} \right| \times_{\Omega^k X_{n+k}} \Omega^k \left| \mathcal{D}_{n+k} \right| \right) \right)_{n \geq 0} \\
&\simeq \left(\left(\operatorname*{colim}_{k \geq 0} \Omega^k \left| \mathcal{C}_{n+k} \right| \right) \times_{X_n} \left(\operatorname*{colim}_{k \geq 0} \Omega^k \left| \mathcal{D}_{n+k} \right| \right) \right)_{n \geq 0} \\
&\simeq |\mathcal{C}| \times_X |\mathcal{D}| .
\end{aligned}$$

To calculate these (co-)limits, we may forget the pointings. The first and last equivalence follow use the given formula and Proposition 3.4.11. The second equivalence holds since $|-|$ preserves pullbacks over spaces by Proposition 3.3.17 and Ω preserves small limits by Proposition 3.4.10. The third equivalence holds since X is an ordinary spectrum and filtered colimits commute with finite limits in **Spc** by [Lur09b, Proposition 5.3.3.3]). □

Now, we establish the precise relation between the theory of categorical spectra and the theory of symmetric monoidal (∞, ∞)-categories.

Proposition 3.4.18. *There a canonical equivalence*

$$\mathbf{CatSp} \simeq \lim \left(\ldots \xrightarrow{\Omega} \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \xrightarrow{\Omega} \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \right).$$

Remark 3.4.19. In particular, each $\Omega^{\infty - n} \colon \mathbf{CatSp} \to \mathbf{Cat}_{(\infty,\infty),*}$ factors through the forgetful functor $\mathbf{Cat}^{\otimes}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,\infty),*}$ as a functor which we also denote $\Omega^{\infty - n} \colon \mathbf{CatSp} \to \mathbf{Cat}^{\otimes}_{(\infty,\infty)}$.

Proof. There is a commutative diagram

$$\begin{array}{ccccc}
 & & \vdots & \vdots & \vdots \\
 & & \downarrow & \downarrow & \downarrow \\
\ldots & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right) & \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{E}_2}\left(\mathbf{Cat}_{(\infty,\infty),*}\right) \\
 & & \downarrow \mathrm{id} & \downarrow & \downarrow \\
\ldots & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{E}_1}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \\
 & & \downarrow \mathrm{id} & \downarrow \mathrm{id} & \downarrow \\
\ldots & \xrightarrow{\Omega} & \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega} \mathbf{Cat}_{(\infty,\infty),*}.
\end{array}$$

Taking the rowwise limit yields the constant diagram $\ldots \to \mathbf{CatSp} \to \mathbf{CatSp}$, whose limit is **CatSp**.

Taking the columnwise limit, on the other hand, yields the desired diagram by [Lur17, Corollary 5.1.1.5]. Thus, the claim follows as limits commute with limits. □

Corollary 3.4.20. *The ∞-category* **CatSp** *admits finite biproducts.*

Proof. This follows from Lemma 3.3.7 and Proposition 3.4.18 since the ∞-category $\mathbf{Cat}^{\otimes}_{(\infty,\infty)}$ admits finite biproducts by Proposition 3.4.3 and these are preserved by $\Omega\colon \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \to \mathbf{Cat}^{\otimes}_{(\infty,\infty)}$. □

Theorem 3.4.21 (Recognition Principle)**.** *There is a canonical adjunction*

$$B^{\infty}\colon \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \rightleftarrows \mathbf{CatSp}\colon \Omega^{\infty}.$$

The left adjoint B^{∞} is given by $\mathcal{C} \mapsto (B^n\mathcal{C})_{n\geq 0}$, is fully faithful, preserves finite products and its essential image consists of the connective categorical spectra, i.e. those $(\mathcal{C}_n)_{n\geq 0}$ for which $\mathcal{C}_n$ is a connective (∞,∞)-category for all $n > 0$.

Proof. This is obtained by applying Corollary B.0.6 to the diagram of Proposition 3.4.18, using the adjunction of Corollary 3.4.7. Since each B^n, $n \geq 0$ preserves finite products, so does B^{∞} by (2) of Propositions 3.4.3 and 3.4.11. It is clear that every categorical spectrum in the essential image of B^{∞} is connective. Conversely, suppose that $(\mathcal{C}_n)_n$ is a connective categorical spectrum. Since $\Omega^k\mathcal{C}_n \simeq \mathcal{C}_{n-k}$ is connective for all $k = 0, \dots, n-1$ by assumption, it follows that $\mathcal{C}_n$ is n-connective for all $n \geq 0$, hence the counit map $B^{\infty}\Omega^{\infty}(\mathcal{C}_n)_n = (B^n\Omega^n\mathcal{C}_n)_n \to (\mathcal{C}_n)_n$ is an equivalence, as desired. □

Now, we explain how this relates to May's classical recognition principle. This discussion illuminates the role of the 'grouplike' condition. Recall that an $\mathbb{E}_1$-monoidal space X is *grouplike* ([Lur17, Definition 5.2.6.6]) if and only the shear maps $(p_1, m), (m, p_2)\colon X \times X \to X \times X$ are equivalences if and only if the monoid $\pi_0(X)$ is a group ([Lur17, Example 5.2.6.4]). In particular, this condition is automatic if X is connected. The Eckmann-Hilton argument implies that, if X is $\mathbb{E}_k$-monoidal ($k \geq 1$), these conditions for any of the k multiplications on X are equivalent to another. The ∞-category $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$ is the full subcategory of $\mathrm{Alg}_{\mathbb{E}_k}(\mathbf{Spc})$ on the grouplike $\mathbb{E}_k$-algebras.

Proposition 3.4.22. *For $k \in \{1, 2, \dots, \infty\}$, the ∞-category $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$ is presentable and the inclusion $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc}) \hookrightarrow \mathrm{Alg}_{\mathbb{E}_k}(\mathbf{Spc})$ admits both adjoints.*

Proof. That $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$ is presentable and the inclusion admits a right adjoint follows by applying [Lur09b, Proposition 5.5.3.13] to the pullback square (of [Lur17, Remark 5.2.6.9])

$$\begin{array}{ccc} \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc}) & \hookrightarrow & \mathrm{Alg}_{\mathbb{E}_k}(\mathbf{Spc}) \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ N(\mathbf{Grp}) & \hookrightarrow & N(\mathbf{Mon}). \end{array}$$

Now, $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$ is the full subcategory of $\mathrm{Alg}_{\mathbb{E}_k}(\mathbf{Spc})$ on those $\mathbb{E}_k$-monoidal spaces X such that the shear maps $(p_1, m), (m, p_2) \colon X \times X \to X \times X$ are equivalences, but since products commute with limits, this subcategory is closed under small limits. Thus, the adjoint functor theorem implies that the inclusion admits a left adjoint $(-)^{\mathrm{gp}} \colon \mathrm{Alg}_{\mathbb{E}_k}(\mathbf{Spc}) \to \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$. □

Theorem 3.4.23. *There are commutative squares ($k \geq 1$)*

$$\begin{array}{ccc} \mathbf{CatSp} & \xrightarrow{\Omega^\infty} & \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \\ \uparrow & & \uparrow \\ \mathbf{Sp} & \xrightarrow{\Omega^\infty} & \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc}), \end{array} \qquad \begin{array}{ccc} \mathbf{Cat}_{(\infty,\infty),*} & \xrightarrow{\Omega^k} & \mathrm{Alg}_{\mathbb{E}_k}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \\ \uparrow & & \uparrow \\ \mathbf{Spc}_* & \xrightarrow{\Omega^k} & \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc}), \end{array} \tag{5}$$

which are horizontally left and vertically right adjointable. Furthermore, we obtain for any $\mathbb{E}_k$-monoidal (∞,∞)-category $\mathcal{C}$ a natural equivalence $\left|B^k\mathcal{C}\right| \simeq B^k |\mathcal{C}|^{\mathrm{gp}}$, $1 \leq k \leq \infty$. In particular, the geometric realization of a k-connective (∞,∞)-category resp. a connective categorical spectrum is a k-connected space resp. a connective spectrum.

Remark 3.4.24. In particular, this describes the right adjoint to $\mathbf{Sp} \hookrightarrow \mathbf{CatSp}$, see also [Ste21, Remark 13.4.20] or [Mas24, Remark 3.4.10].

Proof. For a space X, the $\mathbb{E}_1$-monoidal (∞,∞)-category ΩX is a space by Proposition 3.4.13 and grouplike since the induced monoid structure on $\pi_0(\Omega X)$ is given by identifying $\pi_0(\Omega X) \cong \pi_1(X)$, which is a group. This yields the commutative squares in question.

Now, let X be a grouplike $\mathbb{E}_1$-monoidal space and consider the associated grouplike monoid object $N(\Delta)^{op} \to \mathbf{Spc}$, so that BX is obtained by completing this Segal space object, hence a priori an $(\infty,1)$-category (use Remark 3.1.37). Since Segal equivalences induce equivalences on the (ordinary) homotopy categories, we see that hBX is (up to equivalence) the 1-categorical delooping of the group $\pi_0(X)$, hence a groupoid. Thus, BX is an ∞-groupoid, i.e. a space.

If X was a grouplike $\mathbb{E}_k$-monoidal space for $2 \leq k \leq \infty$, then the induced $\mathbb{E}_{k-1}$-monoidal space BX is again grouplike since it is connected, so we can iterate this procedure and see that B^k carries $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$ into $\mathbf{Spc}_*$ for finite $k \geq 0$ resp. B^∞ carries $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc})$ into $\mathbf{Sp}$. The vertical functors are fully faithful, so this implies horizontal left adjointability by restricting the adjunction.

The right vertical map is the composite $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc}) \hookrightarrow \mathrm{Alg}_{\mathbb{E}_k}(\mathbf{Spc}) \hookrightarrow \mathrm{Alg}_{\mathbb{E}_k}\left(\mathbf{Cat}_{(\infty,\infty)}\right)$. The latter map admits a left adjoint resp. right adjoint induced by the finite product-preserving functors $|-| : \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Spc}$ resp. $(-)^{\simeq} \colon \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Spc}$ and the former admits a left resp. right adjoint by Proposition 3.4.22. Then, vertical right adjointability is a formal consequence of horizontal left adjointability and the last claim is a consequence of taking left adjoints in the original square. □

Corollary 3.4.25 ([Lur17, Theorem 5.2.6.10])**.** *The functor B^∞ induces a fully faithful functor $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc}) \hookrightarrow \mathbf{Sp}$, whose essential image consists precisely of the connective spectra.* □

Remark 3.4.26. In comparison with [Lur17, Theorem 5.2.6.10], this shows that B^∞ restricted to $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathbf{Spc})$ is given by the iterated bar construction.

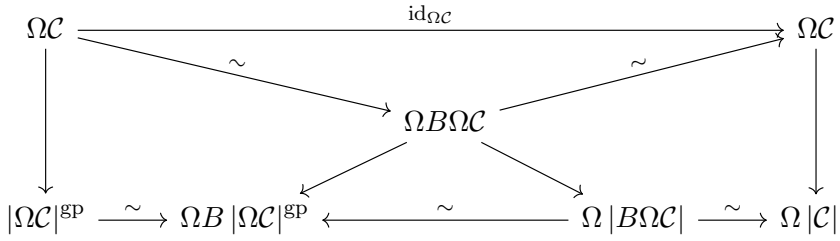
Lastly, we discuss a sufficient condition for the pre-spectrum $(|\mathcal{C}_n|)_{n\geq 0}$ of Remark 3.4.17 to be a spectrum in case that $\mathcal{C}$ is connective.

Definition 3.4.27. A symmetric monoidal (∞,∞)-category $\mathcal{C}$ is *weakly grouplike* if the $\mathbb{E}_\infty$-monoidal space $|\mathcal{C}|$ is grouplike.

Remark 3.4.28. A sufficient condition for $\mathcal{C}$ to be weakly grouplike is that $\mathcal{C}$ (or just its underlying $(\infty,1)$-category) is *rigid* in the sense that every object is dualizable (compare [Lur17, Definition 4.6.1.7, Remark 4.6.1.12]). In that case, the maps $X^\vee \otimes X \to 1$ and $1 \to X \otimes X^\vee$ exhibit the component of $X^\vee$ as a multiplicative inverse to the component of X in $\pi_0(|\mathcal{C}|)$, as desired.

Corollary 3.4.29. *For a connective pointed (∞,∞)-category $\mathcal{C}$, the canonical map $|\Omega\mathcal{C}| \to \Omega\,|\mathcal{C}|$ induces an equivalence $|\Omega\mathcal{C}|^{\mathrm{gp}} \xrightarrow{\sim} \Omega\,|\mathcal{C}|$.*

Proof. This follows by examining the commutative diagram



The outer equivalences are given by the (co-)unit maps of the adjunction of Proposition 3.4.4, using that $\mathcal{C}$ is connective. The upper triangle is one of the triangle identities and the lower triangle is obtained by applying Ω to the triangle $B\Omega\mathcal{C} \to |B\Omega\mathcal{C}| \xrightarrow{\sim} B\,|\Omega\mathcal{C}|^{\mathrm{gp}}$ of Theorem 3.4.23. □

Corollary 3.4.30. *If $\mathcal{C}$ is a weakly grouplike symmetric monoidal (∞,∞)-category, then $|B^\infty\mathcal{C}| \simeq B^\infty\,|\mathcal{C}|$ is obtained as the levelwise geometric realization of $B^\infty\mathcal{C}$.*

Proof. The formula of Corollary 3.4.16 as well as Corollary 3.4.29 imply that this is the case if the $|B^n\mathcal{C}|$ are grouplike, but these are connected for $n > 0$ and grouplike by assumption for $n = 0$. □

Example 3.4.31 (Suspension spectra). The composite $\mathbf{CatSp} \xrightarrow{\Omega^\infty} \mathbf{Cat}^{\otimes}_{(\infty,\infty)} \longrightarrow \mathbf{Cat}_{(\infty,\infty),*}$ admits a left adjoint, given by composing the left adjoint B^∞ of Theorem 3.4.21 and the left adjoint $\mathrm{Free}_{\mathbb{E}_\infty/\mathbb{E}_0}\colon \mathbf{Cat}_{(\infty,\infty),*} \simeq \mathrm{Alg}_{\mathbb{E}_0}\left(\mathbf{Cat}_{(\infty,\infty)}\right) \to \mathbf{Cat}^{\otimes}_{(\infty,\infty)}$ of [Lur17, Corollary 3.1.3.5], the hypotheses of which are satisfied by Proposition 3.3.15. This composite $\Sigma^\infty_{\mathrm{cat}} \coloneqq B^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty/\mathbb{E}_0}$ is

called the *categorical suspension spectrum* functor. It admits an alternative description obtained by applying Corollary B.0.6 to the diagram defining **CatSp**, which gives a natural equivalence $\Sigma^\infty_{\mathrm{cat}} \simeq \left(\mathrm{colim}_{k\geq 0}\, \Omega^k \Sigma^{n+k}_{\mathrm{cat}}\,(-)\right)_{n\geq 0}$ using Proposition 3.4.10. Furthermore, Corollary B.0.8 yields a natural equivalence $\mathcal{C} \simeq \mathrm{colim}_n\, \Sigma^{\infty-n}_{\mathrm{cat}} \mathcal{C}_n$ for any categorical spectrum $\mathcal{C}$, where $\Sigma^{\infty-n}_{\mathrm{cat}} \simeq \Omega^n \Sigma^\infty_{\mathrm{cat}}$ is left adjoint to $\Omega^{\infty-n}$. Thus, every categorical spectrum is a sequential colimit of shifted categorical suspension spectra. The functor $\Sigma^\infty_{\mathrm{cat}}$ does *not* extend the ordinary suspension spectrum functor $\Sigma^\infty \colon \mathbf{Spc}_* \to \mathbf{Sp}$, i.e. the commutative square

$$\begin{array}{ccc} \mathbf{CatSp} & \xrightarrow{\Omega^\infty} & \mathbf{Cat}_{(\infty,\infty),*} \\ \uparrow & & \uparrow \\ \mathbf{Sp} & \xrightarrow{\Omega^\infty} & \mathbf{Spc}_* \end{array}$$

is not horizontally left adjointable (compare Remark 3.4.14). Taking left adjoints, however, yields a natural equivalence $|\Sigma^\infty_{\mathrm{cat}} \mathcal{C}| \simeq \Sigma^\infty\, |\mathcal{C}|$ for any pointed (∞,∞)-category $\mathcal{C}$.

Example 3.4.32 (Finite set spectrum). Interpreting the discrete space S^0 as a pointed (∞,∞)-category yields a categorical suspension spectrum $\mathbb{F} \coloneqq \Sigma^\infty_{\mathrm{cat}} S^0 \simeq B^\infty\, (N(\mathcal{F}\mathrm{in}_*)^{\simeq})$, the *finite set spectrum* ([Mas24, Definition 3.3.11]). This is considered as a higher categorical analogue of the ordinary sphere spectrum $\mathbb{S}$ and, by the previous Example, its geometric realization is indeed $|\mathbb{F}| \simeq \Sigma^\infty S^0 \simeq \mathbb{S}$.

4 Digression: Models of (∞, n)-categories

The notion of a 'model' of (∞,n)-categories was cemented in [BS21]. In this paper, the authors give an axiomatic characterization of the '∞-category of (∞,n)-categories' and prove that this ∞-category is unique up equivalence, in fact unique up to an action of $(\mathbb{Z}/2\mathbb{Z})^n$. Thus, a 'model category of (∞,n)-categories' is any model category presenting this ∞-category ([BS21, Definition 15.4]). In this Section, we introduce a number of such model categories, discuss their equivalence and derive the 'mapping space formula' for geometric realization. The reader is assumed to be familiar with the theory of model categories, see [Hir03] or [Lur09b, Appendix A].

4.1 Complete n-fold Segal Spaces

The category $\mathbf{Cat}_{(\infty,n)}$ as defined in Subsection 3.2 can be presented by a model category by an inductive application of [Lur09a, Proposition 1.5.4]. In this Subsection, we make two such model categories precise. This should be compared to the discussion in [CS19, Part 1, Section 2].

Forthcoming, the category **sSet** of simplicial sets is always equipped with the Kan-Quillen model structure. For any small category $\mathcal{C}$, recall that the category $\mathrm{Fun}\,(\mathcal{C}^{op}, \mathbf{sSet})$ of simplicial presheaves

on $\mathcal{C}$ admits two canonical left proper, combinatorial, simplicial model structures, the *projective* and the *injective* model structure. The injective model structure is cartesian closed, the projective model structure not necessarily, though it is if $\mathcal{C}$ admits finite coproducts. The identity functor determines a Quillen equivalence $\mathrm{id}\colon \mathrm{Fun}\left(\mathcal{C}^{op}, \mathbf{sSet}\right)_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}\left(\mathcal{C}^{op}, \mathbf{sSet}\right)_{\mathrm{inj}} : \mathrm{id}$.

Proposition 4.1.1. *For $n \geq 0$, there is a left proper, combinatorial, simplicial model structure on the category $\mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{sSet}\right)$, denoted $\mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{sSet}\right)_{\mathrm{inj},(\infty,n)}$, which is characterized by the monomorphisms as cofibrations and the injectively fibrant diagrams representing (∞, n)-categories as fibrant objects. The underlying ∞-category of this model category is $\mathbf{Cat}_{(\infty,n)}$.*

Proof. Let $[k_1, \ldots, k_n]$ denote the image of $([k_1], \ldots, [k_n])$ under the Yoneda embedding

$$j\colon \Delta^{\times n} \hookrightarrow \mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{Set}\right) \subseteq \mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{sSet}\right).$$

The 'box product' $\boxtimes\colon \mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{Set}\right) \times \mathrm{Fun}\left(\left(\Delta^{m}\right)^{op}, \mathbf{Set}\right) \to \mathrm{Fun}\left(\left(\Delta^{\times (n+m)}\right)^{op}, \mathbf{Set}\right)$ is given by $(X \boxtimes Y)_{k_1,\ldots,k_{n+m}} = X_{k_1,\ldots,k_n} \times Y_{k_{n+1},\ldots,k_{n+m}}$ for $k_1, \ldots, k_{n+m} \geq 0$. The model structure is obtained by localizing the injective model structure $\mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{sSet}\right)_{\mathrm{inj}}$ with respect to the set S_n of maps

(1) (Segal extensions) $\mathrm{id}_{[k_1,\ldots,k_{i-1}]} \boxtimes \left([1] \sqcup_{[0]} \ldots \sqcup_{[0]} [1] \hookrightarrow [k_i]\right) \boxtimes \mathrm{id}_{[k_{i+1},\ldots,k_n]}$ for $1 \leq i \leq n$ and $k_1, \ldots, k_n \geq 0$,

(2) (Globularity extensions) $[1, \ldots, 1, 0, \ldots, 1, \ldots, 0] \to [1, \ldots, 1, 0, \ldots, 0, \ldots, 0]$ for $1 \leq i \leq n$ and $1 \leq j \leq n - i$ (there are $i - 1$ ones to the left and a one in the $i + j$-th position),

(3) (Completeness extensions) $\mathrm{id}_{[1,\ldots,1]} \boxtimes \left([0] \sqcup_{[1]} [3] \sqcup_{[1]} [0] \to [0]\right) \boxtimes \mathrm{id}_{[0,\ldots,0]}$ for $1 \leq i \leq n$ (there are $i - 1$ ones to the left).

The canonical map $N\left(\mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{sSet}\right)^{\circ}_{\mathrm{inj}}\right) \xrightarrow{\sim} \mathrm{Fun}\left(N\left(\Delta^{\times n}\right)^{op}, \mathbf{Spc}\right)$ induced by the simplicial nerve is an equivalence of ∞-categories by [Lur09b, Proposition 4.2.4.4] and $\mathbf{Cat}_{(\infty,n)}$ is a full subcategory closed under equivalences, so it suffices to prove that an injectively fibrant diagram $X\colon \left(\Delta^{\times n}\right)^{op} \to \mathbf{sSet}$ is S_n-local if and only if it induces an (∞, n)-category $N(X)\colon N\left(\Delta^{\times n}\right)^{op} \to \mathbf{Spc}$ to conclude the proof. To see this, note that an injectively fibrant diagram is local with respect to the Segal extensions if and only if the maps

$$\mathrm{Map}\left([k_1, \ldots, k_i, \ldots, k_n], X\right) \xrightarrow{\sim} \mathrm{Map}\left([k_1, \ldots, k_{i-1}] \boxtimes \left([1] \sqcup_{[0]} \ldots \sqcup_{[0]} [1]\right) \boxtimes [k_i, \ldots, k_n], X\right)$$

are weak equivalences[3]. The Yoneda lemma identifies these with the maps

$$X_{k_1,\ldots,k_n} \xrightarrow{\sim} X_{k_1,\ldots,1,\ldots,k_n} \times_{X_{k_1,\ldots,0,\ldots,k_n}} \ldots \times_{X_{k_1,\ldots,0,\ldots,k_n}} X_{k_1,\ldots,1,\ldots,k_n},$$

[3]Note that every diagram is injectively cofibrant.

but the Yoneda lemma also identifies the face maps $X_{k_1,\dots,1,\dots,k_n} \to X_{k_1,\dots,0,\dots,k_n}$ with the map induced by mapping the injective cofibration $[k_1,\dots,0,\dots,k_n] \hookrightarrow [k_1,\dots,1,\dots,k_n]$ into the injectively fibrant object X, so this map is a fibration. Thus, this fiber product is a homotopy fiber product and this map can be identified by [Lur09b, Theorem 4.2.4.1] with the map

$$N(X)_{k_1,\dots,k_n} \xrightarrow{\sim} N(X)_{k_1,\dots,1,\dots,k_n} \times_{N(X)_{k_1,\dots,0,\dots,k_n}} \cdots \times_{N(X)_{k_1,\dots,0,\dots,k_n}} N(X)_{k_1,\dots,1,\dots,k_n},$$

so we recover the Segal condition for $N(X)$. The globularity and completeness extensions are handled analogously, in the latter case using that $[0] \sqcup_{[1]} [3] \sqcup_{[1]} [0] \cong [3] \sqcup_{[1]\sqcup[1]} ([0] \sqcup [0])$ and the inclusion $[1] \sqcup [1] \hookrightarrow [3]$ identifying the $(0,2)$- and $(1,3)$-edges is an injective cofibration. □

Proposition 4.1.2. *There is a left proper, combinatorial, simplicial model structure on the category* $\mathrm{Fun}\big((\Delta^{\times\infty})^{op}, \mathbf{sSet}\big)$*, denoted* $\mathrm{Fun}\big((\Delta^{\times\infty})^{op}, \mathbf{sSet}\big)_{\mathrm{inj},(\infty,n)}$*, which is characterized by the monomorphisms as cofibrations and the injectively fibrant diagrams representing* (∞,∞)*-categories as fibrant objects. The underlying* ∞*-category of this model category is* $\mathbf{Cat}_{(\infty,\infty)}$*.*

Proof. For every $n \geq 0$, the canonical adjunction $p_n \colon \Delta^{\times\infty} \rightleftarrows \Delta^{\times n} \colon i_n$, where p_n is the projection and i_n fills a tuple up with zeros, induces a simplicial Quillen adjunction

$$p_n^* \colon \mathrm{Fun}\big((\Delta^{\times n})^{op}, \mathbf{sSet}\big)_{\mathrm{inj}} \rightleftarrows \mathrm{Fun}\big((\Delta^{\times\infty})^{op}, \mathbf{sSet}\big)_{\mathrm{inj}} : i_n^*.$$

The model structure is obtained by localizing the injective model structure $\mathrm{Fun}\big((\Delta^{\times\infty})^{op}, \mathbf{sSet}\big)_{\mathrm{inj}}$ with respect to the set of maps $S = S_\infty \coloneqq \bigcup_{n\geq 0} p_n^* S_n$. As in the proof of Proposition 4.1.1, it suffices to prove that an injectively fibrant diagram $X \colon (\Delta^{\times\infty})^{op} \to \mathbf{sSet}$ is S-local if and only if it induces an (∞,∞)-category $N(X) \colon N(\Delta^{\times\infty})^{op} \to \mathbf{Spc}$ to conclude the proof. The adjunction implies that X is S-local if and only if the injectively fibrant diagram $X|_{(\Delta^{\times n})^{op}}$ is S_n-local for all $n \geq 0$. This is equivalent to $N(X)|_{N(\Delta^{\times n})^{op}}$ being an (∞,n)-category for all $n \geq 0$ by Proposition 4.1.1, which is definitionally equivalent to $N(X)$ being an (∞,∞)-category. □

Corollary 4.1.3. *For* $n \in \{0,1,\dots,\infty\}$*, there is a left proper, combinatorial, simplicial model structure on the category* $\mathrm{Fun}\big((\Delta^{\times n})^{op}, \mathbf{sSet}\big)$*, denoted* $\mathrm{Fun}\big((\Delta^{\times n})^{op}, \mathbf{sSet}\big)_{\mathrm{proj},(\infty,n)}$*, which is characterized by the projective cofibrations and the projectively fibrant diagrams representing* (∞,n)*-categories as fibrant objects. The identity functor induces a simplicial Quillen equivalence*

$$\mathrm{id} \colon \mathrm{Fun}\big((\Delta^{\times n})^{op}, \mathbf{sSet}\big)_{\mathrm{proj},(\infty,n)} \rightleftarrows \mathrm{Fun}\big((\Delta^{\times n})^{op}, \mathbf{sSet}\big)_{\mathrm{inj},(\infty,n)} : \mathrm{id}\,.$$

In particular, the underlying ∞*-category of this model category is* $\mathbf{Cat}_{(\infty,n)}$*.*

Proof. This model structure is obtained by instead localizing the projective model structure with

respect to the set of maps S_n of Propositions 4.1.1 or 4.1.2. The commutative diagram

$$\begin{array}{ccc} N\left(\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)^{\circ}_{\mathrm{proj}}\right) & \xrightarrow{\sim} & N\left(\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)^{\circ}_{\mathrm{inj}}\right) \\ \uparrow & \nearrow & \\ N\left(\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)\right) & & \end{array}$$

induced by the Quillen equivalence $\mathrm{id}\colon \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)_{\mathrm{inj}} : \mathrm{id}$ implies that the S_n-local objects and S_n-equivalences, which can be detected in the underlying ∞-category, in the projective and the injective model structure on $\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)$ are the same. The universal property of localization then implies that there is an induced Quillen equivalence $\mathrm{id}\colon \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)_{\mathrm{proj},(\infty,n)} \rightleftarrows \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)_{\mathrm{inj},(\infty,n)} : \mathrm{id}$. □

Remark 4.1.4. For $n \geq 0$, the strongly saturated class generated by S_n agrees with the class $T_{\Delta^{\times n}}$ of [BS21, Remark 14.4], as one sees by examining [BS21, Notation 12.1-12.2].

4.2 Complete Segal Θ_n-spaces

The next model we discuss are Rezk's complete Segal Θ_n-spaces, introduced in [Rez10]. These are based on the idea that the category Θ_n contains all the cells as well as the 'pasting diagrams' that are necessary to express the compositions in an n-category, so it should be possible to conceptualize n-categories as presheaves on this category.

Definition 4.2.1. The category Θ_n is inductively defined as the full subcategory $\mathbf{Cat}^{\mathrm{st}}_n$ spanned by the objects of the form

$$[k|\theta_1, \dots, \theta_k] \coloneqq \Sigma\theta_1 \sqcup_{E_0} \dots \sqcup_{E_0} \Sigma\theta_k, \qquad k \geq 0,\, \theta_1, \dots, \theta_k \in \Theta_{n-1},$$

starting with $\Theta_0 \coloneqq \{*\} \subseteq \mathbf{Set}$. The full subcategory $\Theta = \Theta_\infty \subseteq \mathbf{Cat}^{\mathrm{st}}_\omega$ is defined as the increasing union $\bigcup_{n\geq 0} \Theta_n$ of the $\Theta_n \subseteq \mathbf{Cat}^{\mathrm{st}}_n \subseteq \mathbf{Cat}^{\mathrm{st}}_\omega$ for all $n \geq 0$.

The comparison functor $\delta_n\colon \Delta^{\times n} \to \Theta_n$ is defined by $\delta_0 \coloneqq \mathrm{id}_{\{*\}}$, inductively

$$\delta_n\left([k_1], \dots, [k_n]\right) \coloneqq [k_1|\delta_{n-1}([k_2], \dots, [k_n]), \dots, \delta_{n-1}([k_2], \dots, [k_n])]\,, \qquad k_1, \dots, k_n \geq 0,$$

and $\delta_\infty \coloneqq \bigcup_{n\geq 1} \delta_n$.

Lemma 4.2.2. *The functors δ_n, $n \geq 0$ satisfy the following properties:*

(i) For $k_1, \dots, k_{n-1} \geq 0$, we have $\delta_n\left([1], [k_1], \dots, [k_{n-1}]\right) = \Sigma\delta_{n-1}\left([k_1], \dots, [k_{n-1}]\right)$.

(ii) *For* $k_1, \dots, k_{n-1} \geq 0$, *we have* $\delta_n([k_1], \dots, [k_{i-1}], [0], [k_i], \dots, [k_{n-1}]) = \delta_{i-1}([k_1], \dots, [k_{i-1}])$.

(iii) *The functor* $\delta_n \colon \Delta^{\times n} \to \Theta_n$ *preserves pushouts of the form* $[1] \sqcup_{[0]} \dots \sqcup_{[0]} [1] \xrightarrow{\sim} [k]$ *in each variable separately.*

(iv) *For* $k_1, \dots, k_{n+1} \geq 0$, *we have* $(\pi_0)_n \delta_{n+1}([k_1], \dots, [k_{n+1}]) = \delta_n([k_1], \dots, [k_n])$.

Proof. The identity (i) is immediate by definition. The identity (ii) reduces by an easy induction to the observation that $\delta_n([0], [k_1], \dots, [k_{n-1}]) = E_0$ for any $k_1, \dots, k_{n-1} \geq 0$. The property (iii) reduces by an induction, using that Σ that preserves pushouts (see Definition 2.2.5), to preservation in the first variable, which is again clear by definition. The property (iv) is yet another easy induction. □

Definition 4.2.3 ([OR23, Definition 4.13],[Lou23, Definition 3.4.3.3])**.** For $n \in \{0, 1, \dots, \infty\}$, let $j \colon \Theta_n \to \mathrm{Fun}(\Theta_n^{op}, \mathbf{Set}) \subseteq \mathrm{Fun}(\Theta_n^{op}, \mathbf{sSet})$ denote the Yoneda embedding. A presheaf $X \colon \Theta_n^{op} \to \mathbf{sSet}$ is a *complete Segal* Θ_n*-space* if and only if it is local with respect to the set T_n of maps

(1) (m-fold k-Segality extensions) For $1 \leq m \leq n$, $k \geq 1$ and $\theta_1, \dots, \theta_k \in \Theta_{n-m}$, the map

$$j(\Sigma^m \theta_1) \sqcup_{j(\Sigma^{m-1} E_0)} \dots \sqcup_{j(\Sigma^{m-1} E_0)} j(\Sigma^m \theta_k) \to j\left(\Sigma^{m-1}[k|\theta_1, \dots, \theta_k]\right).$$

(2) (m-fold completeness extensions) For $0 \leq m \leq n-1$, the map

$$j(\Sigma^m[0]) \to j(\Sigma^m[0]) \sqcup_{j(\Sigma^m[1])} j(\Sigma^m[3]) \sqcup_{j(\Sigma^m[1])} j(\Sigma^m[0]).$$

Theorem 4.2.4 ([OR23, Theorem 4.14],[Lou23, Definition 3.4.3.3])**.** *There is a cartesian closed, left proper, combinatorial, simplicial model structure on* $\mathrm{Fun}(\Theta_n^{op}, \mathbf{sSet})$ *(a localization of the projective model structure), denoted* $\mathrm{Fun}(\Theta_n^{op}, \mathbf{sSet})_{\mathrm{proj},(\infty,n)}$. *This model structure is characterized by the projective cofibrations and the projectively fibrant complete Segal* Θ_n*-spaces as its fibrant objects.* □

Remark 4.2.5. Though [OR23, Definition 4.13] and [Lou23, Definition 3.4.3.3] use different generating sets, one can check that they generate the same strongly saturated class. The model structure constructed here differs from the one in [Rez10], which is obtained by instead localizing the injective model structure, as explained in [OR23, Remark 4.15]. As in Corollary 4.1.3, either presents the underlying ∞-category $CSS(\Theta_n) \subseteq \mathrm{Fun}(N(\Theta_n)^{op}, \mathbf{Spc})$, which is the full subcategory on those functors $N(\Theta_n)^{op} \to \mathbf{Spc}$ that are local with respect to (cofibrant replacements of) the above extensions. This is the ∞-category discussed in [BS21, Section 13], as explained there.

Proposition 4.2.6. *For any* $n \geq 0$, *the functor* $\delta_n \colon \Delta^{\times n} \to \Theta_n$ *induces a Quillen equivalence*

$$(\delta_n)_! \colon \mathrm{Fun}\left(\left(\Delta^{\times n}\right)^{op}, \mathbf{sSet}\right)_{\mathrm{proj},(\infty,n)} \rightleftarrows \mathrm{Fun}(\Theta_n^{op}, \mathbf{sSet})_{\mathrm{proj},(\infty,n)} : \delta_n^*.$$

Proof. First of all, $(\delta_n)_!\colon \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}\left(\Theta_n^{op}, \mathbf{sSet}\right)_{\mathrm{proj}} : \delta_n^*$ is a Quillen adjunction by [Lur09b, Proposition A.2.8.7, (1)]. To show that this induces a Quillen adjunction between the respective localizations, it suffices to show that $(\delta_n)_!$ maps S_n into the strongly saturated class generated by T_n, by the universal property of localization. This is established as part of [BS21, Theorem 14.6]. The right Quillen functor induces an equivalence of underlying ∞-categories by [BS21, Corollary 14.7], so this Quillen adjunction is in particular a Quillen equivalence. □

Remark 4.2.7. For the localizations of the *injective* model structures, this is [BS21, Example 15.12].

Theorem 4.2.8. *The functor $\delta_\infty\colon \Delta^{\times\infty} \to \Theta$ induces a Quillen equivalence*

$$(\delta_\infty)_!\colon \mathrm{Fun}((\Delta^{\times\infty})^{op}, \mathbf{sSet})_{\mathrm{proj},(\infty,n)} \rightleftarrows \mathrm{Fun}(\Theta^{op}, \mathbf{sSet})_{\mathrm{proj},(\infty,n)}\colon \delta_\infty^*.$$

Proof. First of all, $(\delta_\infty)_!\colon \mathrm{Fun}((\Delta^{\times\infty})^{op}, \mathbf{sSet})_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}(\Theta^{op}, \mathbf{sSet})_{\mathrm{proj}}\colon \delta_\infty^*$ is a Quillen adjunction by [Lur09b, Proposition A.2.8.7, (1)]. For any $n \geq 0$, there is a commutative square

$$\begin{array}{ccc} \Delta^{\times n} & \xhookrightarrow{i_n} & \Delta^{\times\infty} \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_\infty} \\ \Theta_n & \xhookrightarrow{i'_n} & \Theta. \end{array}$$

Furthermore, as explained in the proof of Proposition 4.1.2, we have $(i_n)_! = p_n^*$. Thus,

$$(\delta_\infty)_!(S) = \bigcup_{n\geq 0} (\delta_\infty)_! p_n^* S_n = \bigcup_{n\geq 0} (\delta_\infty)_! (i_n)_! S_n = \bigcup_{n\geq 0} (i'_n)_! (\delta_n)_! S_n.$$

As explained in the proof of Proposition 4.2.6, $(\delta_n)_! S_n$ is contained in the strongly saturated class generated by T_n. Furthermore, $(i'_n)_!$ preserves colimits and representables, so it takes the strongly saturated generated by T_n into the strongly saturated class generated by T_∞. Thus, the universal property implies that we obtain an induced Quillen adjunction between the respective localizations.

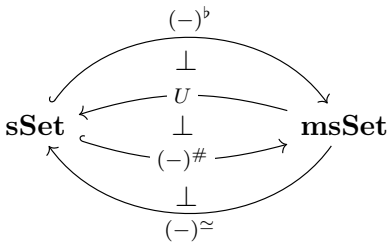
To show that this Quillen adjunction is a Quillen equivalence, it suffices to prove that the right Quillen functor δ_∞^* induces an equivalence $CSS(\Theta) \to \mathbf{Cat}_{(\infty,\infty)}$ of the underlying ∞-categories. Since δ_n^* induces compatible equivalences $CSS(\Theta_n) \xrightarrow{\sim} \mathbf{Cat}_{(\infty,n)}$ for any $n \geq 0$ and $\mathbf{Cat}_{(\infty,\infty)} = \lim_{n\geq 0} \mathbf{Cat}_{(\infty,n)}$, it suffices to show that $CSS(\Theta) = \lim_{n\geq 0} CSS(\Theta_n)$. This holds since a diagram $X\colon \Theta^{op} \to \mathbf{sSet}$ is T_∞-local if and only if $X|_{\Theta_n^{op}}$ is T_n-local for every $n \geq 0$, as every element of T_∞ is in the image of some $(i'_n)_!$, $n \geq 0$ by inspection. □

4.3 Complicial Sets

The last model we discuss are the *complicial sets*, originally introduced by Roberts and Street, later refined by Verity and Riehl. These are a generalization of Joyal's quasicategories. If we want to incarnate (∞, n)-categories as simplicial sets, the issue emerges that n-simplices for $n \geq 2$ serve a two-fold purpose: they are representatives of n-morphisms as well as witnesses to the compositions of $(n-1)$-morphisms, the latter of which are 'invertible'. The idea is to 'mark' the latter to begin with.

Definition 4.3.1 ([Ver05, Definition 96])**.** A *marked (or stratified) simplicial set* is a pair (X, tX) consisting of a simplicial set X and subsets $tX_n \subseteq X_n$, $n \geq 1$ including all degenerate simplices, whose elements are called *marked simplices*. These are the objects in a category **msSet**, whose morphisms are the simplicial maps that preserve marked simplices.

Proposition 4.3.2. *There is an adjoint quadruple*



The functors $(-)^\flat$ resp. $(-)^\#$ equip a simplicial set with the minimal resp. maximal marking, U is the obvious forgetful functor and $(-)^\simeq$ takes the simplicial subset on simplices whose faces in all dimensions are marked. □

These marked simplicial sets can also be used to model the theory of strict ω-categories.

Definition 4.3.3 ([Str87, Section 5])**.** The unique functor $N_{\mathcal{O}}^{RS} \colon \mathbf{Cat}_\omega^{\mathrm{st}} \hookrightarrow \mathbf{msSet}$ such that $UN_{\mathcal{O}}^{RS} = N_{\mathcal{O}}$ and an n-simplex of $N_{\mathcal{O}}(\mathcal{C})$ is marked if and only if the corresponding functor $\mathcal{O}[n] \to \mathcal{C}$ sends the unique n-dimensional atom to an identity n-morphism in $\mathcal{C}$ is the *Roberts-Street nerve*.

For a strict ω-category $\mathcal{C}$, the marked simplicial set $N_{\mathcal{O}}^{RS}(\mathcal{C})$ can be understood as a model for the strict ω-category $\mathcal{C}$ itself, in the sense of the following Theorem.

Theorem 4.3.4 ([Str87, Conjecture 5.3],[Ver05, Theorem 266])**.** *The Roberts-Street nerve*

$$N_{\mathcal{O}}^{RS} \colon \mathbf{Cat}_\omega^{\mathrm{st}} \hookrightarrow \mathbf{msSet}$$

is fully faithful. □

Remark 4.3.5. Furthermore, the Theorem characterizes the essential image of this functor as the *strict complicial sets*, which however do not play a role in this Thesis.

Definition 4.3.6 ([Ver08, Definition 8])**.** A monomorphism $i\colon X \hookrightarrow Y$ of marked simplicial sets is *regular* if $tX_n = i^{-1}(tY_n)$ for all $n \geq 1$, and *entire* if $U(i)$ is an identity morphism.

Definition 4.3.7 ([Ver08, Definition 31])**.** The *join* $X \star Y$ of two marked simplicial sets X, Y is given by taking the join of underlying simplicial sets and marking a simplex $\sigma \star \tau$ of $X \star Y$ if and only if σ is marked in X or τ is marked in Y.

Definition 4.3.8 ([OR20, Definition 1.19])**.** Let $n \in \{0, 1, \dots, \infty\}$. The *$n$-complicial anodyne extensions* are the

- (complicial horn extensions) For $m \geq 1$ and $0 \leq k \leq m$, the regular inclusion $\Lambda^m_k \hookrightarrow \Delta^m_k$, where Δ^m_k is $(\Delta^m)^\flat$ with the simplices containing $\{k-1, k, k+1\} \cap [m]$ additionally marked.
- (complicial thinness extensions) For $m \geq 2$ and $0 \leq k \leq m$, the entire inclusion $(\Delta^m_k)' \hookrightarrow (\Delta^m_k)''$, where $(\Delta^m_k)'$ is obtained from Δ^m_k by marking the $(k-1)$-th and $(k+1)$-th face, if they exist, and $(\Delta^m_k)''$ is obtained from $(\Delta^m_k)'$ by further marking the k-th face.
- (triviality extensions) For $m \geq n$, the entire inclusions $(\Delta^m)^\flat \hookrightarrow (\Delta^m)_t$, where $(\Delta^m)_t$ is $(\Delta^m)^\flat$ with the top simplex additionally marked.
- (saturation extensions) For $l, m \geq -1$, the entire inclusions $\left(\Delta^l\right)^\flat \star \Delta^3_{eq} \star (\Delta^m)^\flat \hookrightarrow \left(\Delta^l\right)^\flat \star (\Delta^3)^\# \star (\Delta^m)^\flat$, where Δ^3_{eq} is $\left(\Delta^3\right)^\flat$ with the edges $(0,2)$ and $(1,3)$ as well as all simplices in dimension ≥ 2 additionally marked.

Definition 4.3.9 ([OR20, Definition 1.21])**.** An *n-complicial set* is a marked simplicial set that has the right lifting property with respect to the n-complicial anodyne extensions.

Remark 4.3.10. The marked simplicial sets that have the right lifting property with respect to the first three classes are called 'n-trivial weak complicial sets' in [Ver08, Example 102]. The saturation condition was introduced in [Ver17] and [Rie18], see also [OR21, Digression 2.1] for a discussion. The motivation for these lifting properties is extensively detailed in [Rie18].

Remark 4.3.11. The 0-complicial sets are the Kan complexes with the maximal marking. The 1-complicial sets are the quasicategories with the equivalences and all simplices in dimension ≥ 2 marked.

Theorem 4.3.12 ([OR20, Theorem 1.25])**.** *For $n \in \{0, 1, \dots, \infty\}$, there is a cartesian closed, left proper, combinatorial model structure on* **msSet***, denoted* $\mathbf{msSet}_{(\infty,n)}$*, which is characterized by the monomorphisms as cofibrations and the n-complicial sets as fibrant objects.* □

Remark 4.3.13. The existence of this model structure was originally claimed in [Rie18, Example 3.3.6].

Remark 4.3.14. The extensions of Definition 4.3.9 are acyclic cofibrations in $\mathbf{msSet}_{(\infty,n)}$, but they do not form a generating set of acyclic cofibrations.

Lemma 4.3.15 ([OR23, Proposition 4.10])**.** *For $n \in \{0, 1, \dots, \infty\}$, there is a Quillen adjunction*

$$(-)^{\#} \colon \mathbf{sSet} \rightleftarrows \mathbf{msSet}_{(\infty,n)} \colon (-)^{\simeq}.$$

This is a Quillen equivalence if $n = 0$. □

Corollary 4.3.16. *For $n \in \{0, 1, \dots, \infty\}$. the cartesian closed model category $\mathbf{msSet}_{(\infty,n)}$ is a simplicial model category with respect to the change of enrichment along $(-)^{\simeq} : \mathbf{msSet} \to \mathbf{sSet}$.*

Proof. The category is tensored over $\mathbf{sSet}$ by $X \otimes Y \coloneqq X^{\#} \times Y$ for $X \in \mathbf{sSet}$, $Y \in \mathbf{msSet}$. Indeed, we calculate for any simplicial set K (here, $\mathrm{Map}\,(-,-)$ denotes the either internal Hom)

$$\begin{aligned}
\mathbf{sSet}\left(K, \mathrm{Map}\,(X \otimes Y, Z)^{\simeq}\right) &\cong \mathbf{msSet}\left(K^{\#}, \mathrm{Map}\left(X^{\#} \times Y, Z\right)\right) \cong \mathbf{msSet}\left(K^{\#} \times X^{\#} \times Y, Z\right) \\
&\cong \mathbf{msSet}\left((K \times X)^{\#} \times Y, Z\right) \cong \mathbf{msSet}\left((K \times X)^{\#}, \mathrm{Map}\,(Y, Z)\right) \\
&\cong \mathbf{sSet}\left(K \times X, \mathrm{Map}\,(Y, Z)^{\simeq}\right) \cong \mathbf{sSet}\left(K, \mathrm{Map}\left(X, \mathrm{Map}\,(Y, Z)^{\simeq}\right)\right).
\end{aligned}$$

Thus, the Yoneda lemma yields a natural isomorphism $\mathrm{Map}\,(X \otimes Y, Z)^{\simeq} \cong \mathrm{Map}\left(X, \mathrm{Map}\,(Y, Z)^{\simeq}\right)$. A similar argument shows the category is cotensored over $\mathbf{sSet}$ by $Y^X \coloneqq \mathrm{Map}\left(X^{\#}, Y\right)$.

The model category $\mathbf{msSet}_{(\infty,n)}$ is cartesian closed by Theorem 4.3.12 and $(-)^{\#}$ is a left Quillen functor by Lemma 4.3.15, hence the tensoring

$$\mathbf{sSet} \times \mathbf{msSet}_{(\infty,n)} \xrightarrow{(-)^{\#} \times \mathrm{id}} \mathbf{msSet}_{(\infty,n)} \times \mathbf{msSet}_{(\infty,n)} \xrightarrow{(-)\times(-)} \mathbf{msSet}_{(\infty,n)}$$

is a left Quillen bifunctor. □

Remark 4.3.17. This proof shows, more generally, that given a right Quillen functor $R \colon \mathcal{C} \to \mathcal{D}$ of cartesian closed model categories whose left adjoint preserves finite products, the change of self-enrichment along R makes $\mathcal{C}$ into a $\mathcal{D}$-enriched model category.

Theorem 4.3.18 ([OR23, Theorem 4.16])**.** *For $n \in \{0, \dots, \infty\}$, there is a Quillen adjunction*

$$L_n \colon \mathrm{Fun}\,(\Theta_n^{op}, \mathbf{sSet})_{\mathrm{proj},(\infty,n)} \rightleftarrows \mathbf{msSet}_{(\infty,n)} \colon R_n.$$

The right adjoint is given by $R_n(X)\,(\Theta) = \mathbf{msSet}\left(N_{\mathcal{O}}^{RS}\,(\Theta) \times (\Delta^{\bullet})^{\#}, X\right)$ for $X \in \mathbf{msSet}$, $\theta \in \Theta_n$. □

Remark 4.3.19. The marked simplicial maps $N^{RS}_{\mathcal{O}}(\Theta) \times (\Delta^m)^{\#} \to X$ correspond by adjunction to simplicial maps $\Delta^m \to \mathrm{Map}\left(N^{RS}_{\mathcal{O}}(\Theta), X\right)^{\simeq}$, so we have $R_n(X)(\Theta) = \mathrm{Map}\left(N^{RS}_{\mathcal{O}}(\Theta), X\right)^{\simeq}$.

Theorem 4.3.20 ([Lou23, Theorem 3.4.3.14])**.** *The Quillen adjunction of Theorem 4.3.18 is a Quillen equivalence.* □

Lastly, we present a variant of this model analogous to complete Segal spaces.

Definition 4.3.21. The category $t\Delta$ has objects $[n]$, $n \geq 0$ and $[n]_t$, $n \geq 1$, and the morphisms are determined by specifying that $t\Delta \hookrightarrow \mathbf{msSet}$, $[n] \mapsto \Delta^n$, $[n]_t \mapsto (\Delta^n)_t$ is a fully faithful functor. This determines an adjunction $R\colon \mathrm{Fun}(t\Delta^{op}, \mathbf{Set}) \rightleftarrows \mathbf{msSet}\colon i$, whose right adjoint i is fully faithful and its essential image consists of those presheaves $X\colon t\Delta^{op} \to \mathbf{Set}$ such that $X([n]_t) \to X([n])$ induced by $[n] \to [n]_t$ is a monomorphism for all $n \geq 1$. The left adjoint R is given by restricting along $\Delta \hookrightarrow t\Delta$ and marking those simplices in the image of $X([n]_t) \to X([n])$ for all $n \geq 1$.

Proposition 4.3.22 ([OR20, Theorem 1.28])**.** *For $n \in \{0, 1, \ldots, \infty\}$, there is a cartesian closed, left proper, combinatorial model structure on* $\mathrm{Fun}(t\Delta^{op}, \mathbf{Set})$*, denoted* $\mathrm{Fun}(t\Delta^{op}, \mathbf{Set})_{(\infty,n)}$*, which is characterized by the monomorphisms as cofibrations and the presheaves with the right lifting property against the images of the n-complicial extensions as fibrant objects.* □

Proposition 4.3.23 ([OR20, Proposition 1.35])**.** *For $n \in \{0, 1, \ldots, \infty\}$, the adjunction*

$$R\colon \mathrm{Fun}(t\Delta^{op}, \mathbf{Set})_{(\infty,n)} \rightleftarrows \mathbf{msSet}_{(\infty,n)}\colon i$$

is a Quillen equivalence. □

Theorem 4.3.24 ([OR20, Theorem 2.7])**.** *For $n \in \{0, 1, \ldots, \infty\}$, there is a cartesian closed, left proper, combinatorial model structure on* $\mathrm{Fun}(t\Delta^{op}, \mathbf{sSet})$ *(a localization of the injective model structure), denoted* $\mathrm{Fun}(t\Delta^{op}, \mathbf{sSet})_{\mathrm{inj},(\infty,n)}$*, which is characterized by the monomorphisms as cofibrations and the simplicial presheaves that are injectively fibrant and local with respect to the images of the n-complicial anodyne extensions as fibrant objects.* □

Theorem 4.3.25 ([OR20, Theorem 2.14])**.** *For $n \in \{0, 1, \ldots, \infty\}$, the 'nerve-realization' adjunction* $\mathrm{Real}\colon \mathrm{Fun}(t\Delta^{op}, \mathbf{sSet})_{\mathrm{inj},(\infty,n)} \rightleftarrows \mathrm{Fun}(t\Delta^{op}, \mathbf{Set})_{(\infty,n)} : \mathrm{Sing}$ *induced by*

$$t\Delta \times \Delta \to \mathrm{Fun}(t\Delta^{op}, \mathbf{Set}),\ ([n], [m]) \mapsto (\Delta^n)^{\flat} \times (\Delta^m)^{\#},\ ([n]_t, [m]) \mapsto (\Delta^n)_t \times (\Delta^m)^{\#}$$

is a Quillen equivalence. □

Corollary 4.3.26. *For $n \in \{0, 1, \ldots, \infty\}$, there is a Quillen equivalence*

$$R \circ \mathrm{Real}\colon \mathrm{Fun}(t\Delta^{op}, \mathbf{sSet})_{\mathrm{inj},(\infty,n)} \rightleftarrows \mathbf{msSet}_{(\infty,n)}\colon \mathrm{Sing} \circ i,$$

where $\mathrm{Sing} \circ i$ *takes a marked simplicial set* X *to the simplicial presheaf on* $t\Delta$ *given by*

$$[n] \mapsto \mathrm{Map}\left((\Delta^n)^\flat, X\right)^{\simeq}, \qquad [n]_t \mapsto \mathrm{Map}\left((\Delta^n)^\# , X\right)^{\simeq}. \qquad \square$$

4.4 The Nerve Construction

It should be a litmus test for our theory of weak higher categories that any strict n-category gives rise to an (∞, n)-category. It turns out, however, that this is more subtle than one might expect.

To begin, we have exhibited $\mathbf{Cat}_n^{\mathrm{st}}$ as a reflective subcategory of $\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{Set}\right)$ in the proof of Theorem 2.2.7. Thus, our first guess is to take an n-category $\mathcal{C}$ and interpret it as a *discrete* n-fold simplicial space. To study this, we have to make the embedding $\mathbf{Cat}_n^{\mathrm{st}} \hookrightarrow \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{Set}\right)$ more explicit. The next discussion is to a large extent implicit in [BS21, Section 14].

Proposition 4.4.1. *The adjunction* $\mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{Set}\right) \rightleftarrows \mathbf{Cat}_n^{\mathrm{st}}$ *constructed in the proof of Theorem 2.2.7 is induced by the* n*-fold cosimplicial object* $\Delta^{\times n} \xrightarrow{\delta_n} \Theta_n \subseteq \mathbf{Cat}_n^{\mathrm{st}}$.

Proof. This is proven by induction on n, the base case being vacuous. In the inductive step, we need to identify the composite

$$\begin{aligned}\Delta^{\times n} \hookrightarrow \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{Set}\right) = \mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{sSet}\right) \xrightarrow{h_*} \mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{Cat}\right) \\ \longrightarrow \mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{Set}\right) \text{-} \mathbf{Cat} \longrightarrow \mathbf{Cat}_{n-1}^{\mathrm{st}} \text{-} \mathbf{Cat} = \mathbf{Cat}_n^{\mathrm{st}}.\end{aligned}$$

This takes an object $([k_1], \ldots, [k_n]) \in \Delta^{\times n}$ to the $(n-1)$-fold simplicial object in $\mathbf{Cat}$ given by $[k_1] \times D\Delta^{\times(n-1)}(-, ([k_2], \ldots, [k_n]))$. The induces by the formula (1) the $(n-1)$-fold simplicial object

$$\left([k_1] \times \Delta^{\times(n-1)}(-, ([k_2], \ldots, [k_n]))\right) \sqcup_{\mathrm{Ob}([k_1]) \times \Delta^{\times(n-1)}(-, ([k_2], \ldots, [k_n]))} \mathrm{Ob}([k_1])$$

in $\mathbf{Cat}$, which corresponds to the $\mathrm{Fun}\left(\left(\Delta^{\times(n-1)}\right)^{op}, \mathbf{Set}\right)$-enriched category that has objects $0, \ldots, k_1$ and is freely generated by $\Delta^{\times(n-1)}(-, ([k_2], \ldots, [k_n]))$ as the $(n-1)$-fold simplicial set of morphisms $i-1 \to i$ for any $1 \leq i \leq k_1$. Now, applying the inductive assumption, this maps to

$$\Sigma\delta_{n-1}([k_2], \ldots, [k_n]) \sqcup_{E_0} \ldots \sqcup_{E_0} \Sigma\delta_{n-1}([k_2], \ldots, [k_n]) = \delta_n([k_1], \ldots, [k_n]). \qquad \square$$

The following definition will become relevant soon.

Definition 4.4.2 ([BS21, Definition 3.1])**.** For any $n \geq 0$, a strict n-category $\mathcal{C}$ is *gaunt* if every invertible morphism in $\mathcal{C}$ is an identity. The category Gaunt_n is the full subcategory of $\mathbf{Cat}_n^{\mathrm{st}}$ on the gaunt n-categories. The category Gaunt_ω of gaunt ω-categories is $\mathrm{Gaunt}_\omega \coloneqq \bigcup_{n \geq 0} \mathrm{Gaunt}_n \subseteq \mathbf{Cat}_\omega^{\mathrm{st}}$.

There are plenty of examples of gaunt categories.

Proposition 4.4.3. *For any $n \geq 0$, $\Theta_n \subseteq \mathrm{Gaunt}_n$. It follows that $\Theta \subseteq \mathrm{Gaunt}_\omega$.*

Proof. This is proven by induction on n, the base case being vacuous. In the inductive step, an object of Θ_n has the form $[k|\theta_1, \dots, \theta_k] = \Sigma\theta_1 \sqcup_{E_0} \dots \sqcup_{E_0} \Sigma\theta_k$ for some $k \geq 0$ and $\theta_1, \dots, \theta_k \in \Theta_{n-1}$. In particular, any non-identity 1-morphism in this category has the form $i \to j$ for some $0 \leq i < j \leq k$, hence isn't invertible. The invertibility of a k-morphisms for $k \geq 2$ is decided in a Hom-category, but $[k|\theta_1, \dots, \theta_k](i,j) = \theta_{i+1} \times \dots \times \theta_j$ for $0 \leq i < j \leq k$ is gaunt by the inductive hypothesis. □

Proposition 4.4.4. *For any $n \geq 0$, let C be an augmented directed chain complex that is concentrated in degrees up to n and has a basis. Then, νC is a gaunt n-category.*

Proof. It is clear by definition that νC is a strict n-category. Let $x = \left(x_k^\pm | \dots | x_0^\pm\right)$, $y = \left(y_k^\pm | \dots | y_0^\pm\right)$ be two inverse k-morphisms in νC, $1 \leq k \leq n$. These are composable in either order, hence $x_{k-1}^+ = y_{k-1}^-$, $y_{k-1}^+ = x_{k-1}^-$ and $x_l^\pm = y_l^\pm$ for $0 \leq l \leq k-2$. Then,

$$\left(0, 0 | x_{k-1}^-, x_{k-1}^- | x_{k-2}^\pm | \dots | x_0^\pm\right) = 1_{s(x)} = y \circ_{k-1} x = \left(y_k^\pm + x_k^\pm | x_{k-1}^-, x_{k-1}^- | x_{k-2}^\pm | \dots | x_0^\pm\right),$$

hence $y_k^\pm + x_k^\pm = 0$, i.e. $y_k^\pm = -x_k^\pm$. Since $x_k^\pm, y_k^\pm \in C_k^+$ both have positive coefficients in a basis expansion, this implies $x_k^\pm = y_k^\pm = 0$. In particular, $x_{k-1}^+ - x_{k-1}^- = \partial\left(x_k^\pm\right) = 0$, hence $x_{k-1}^- = x_{k-1}^+$ and $x = 1_{\left(x_{k-1}^-, x_{k-1}^- | x_{k-2}^\pm | \dots | x_0^\pm\right)}$. □

Corollary 4.4.5. *For any $n \geq 0$, the n-th oriental $\mathcal{O}[n]$ is a gaunt n-category.* □

Let us also recall the adjunction $\pi_0 \colon \mathbf{Spc} \rightleftarrows N(\mathbf{Set}) \colon c$ of the connected component and constant simplicial set functor. Its couint is an isomorphism, hence $c \colon N(\mathbf{Set}) \hookrightarrow \mathbf{Spc}$ is fully faithful.

Lemma 4.4.6. *Let $\mathcal{C}$ be 1-category. The discrete simplicial space $c_* N(\mathcal{C})$ is a complete Segal space if and only if $\mathcal{C}$ is gaunt.*

Proof. The right adjoint c preserves finite limits, hence $c_* N(\mathcal{C})$ is a category object in $\mathbf{Spc}$ by Proposition 3.1.11. Its underlying homotopy category is $\mathcal{C}$ itself, so $(c_* N(\mathcal{C}))^\sim \cong c_* N(\mathcal{C}^\simeq)$ and this simplicial space is constant if and only if every morphism in $\mathcal{C}^\simeq$ is an identity, i.e. $\mathcal{C}$ is gaunt. □

Proposition 4.4.7. *For any $n \geq 0$ and strict n-category $\mathcal{C}$, the discrete n-fold simplicial space*

$$X \colon \Delta^{\times n} \to \mathbf{Spc},\ ([k_1], \dots, [k_n]) \mapsto c\mathbf{Cat}_n^{\mathrm{st}}\left(\delta_n([k_1], \dots, [k_n]), \mathcal{C}\right)$$

is an n-fold Segal space. It is complete if and only if $\mathcal{C}$ is gaunt.

Proof. By Lemma 4.2.2,(iii), δ_n preserves pushouts of the form $[1]\sqcup_{[0]}\ldots\sqcup_{[0]}[1] \xrightarrow{\sim} [k]$ in each variable separately, which translates under the limit-preserving functor $c\mathbf{Cat}_n^{\mathrm{st}}(-,\mathcal{C})$ to the Segal conditions. The globularity conditions are a direct consequence of Lemma 4.2.2,(ii). Now, for $0 \leq i \leq n$, the (discrete) simplicial space $X_{1,\ldots,1,\bullet,0,\ldots,0}$ is given by

$$[n] \mapsto c\mathbf{Cat}_n^{\mathrm{st}}\left(\delta_n\left([1],\ldots,[1],[n],[0],\ldots,[0]\right),\mathcal{C}\right).$$

Lemma 4.2.2,(i) identifies $\delta_n\left([1],\ldots,[1],[n],[0],\ldots,[0]\right) = \Sigma^{i-1}[n]$, hence the n-simplices correspond to sequences of n composable i-morphisms in $\mathcal{C}$, i.e. $X_{1,\ldots,1,\bullet,0,\ldots,0}$ is the nerve of the 1-category whose objects are the $(i-1)$-morphisms and whose morphisms are the i-morphisms of $\mathcal{C}$. This category is gaunt if and only if every invertible i-morphisms in $\mathcal{C}$ is an identity. Thus, X is complete if and only if $\mathcal{C}$ is gaunt by applying Lemma 4.4.6. □

Corollary 4.4.8. *For any $n \geq 0$, the composite*

$$N(\mathbf{Cat}_n^{\mathrm{st}}) \hookrightarrow \mathrm{Fun}\left(N\left(\Delta^{\times n}\right)^{op}, N(\mathbf{Set})\right) \xrightarrow{c_*} \mathrm{Fun}\left(N\left(\Delta^{\times n}\right)^{op}, \mathbf{Spc}\right)$$

restricts to a fully faithful functor $N_n\colon N(\mathrm{Gaunt}_n) \hookrightarrow \mathbf{Cat}_{(\infty,n)}$.

Proof. Theorem 2.2.7 and Proposition 4.4.1 imply that the composite is fully faithful and Proposition 4.4.7 implies that it restricts to a functor $N(\mathrm{Gaunt}_n) \hookrightarrow \mathbf{Cat}_{(\infty,n)}$. □

Remark 4.4.9. It can be shown that the essential image of this functor are precisely the discrete n-fold complete Segal spaces.

To extend this nerve construction to not necessarily gaunt categories, we use the model of complicial sets. To this end, we present another variation of the Street nerve.

Definition 4.4.10 ([OR24, Definitions 1.5.1-2])**.** Let $\mathcal{C}$ be a strict ω-category. A *structure of reversibility* in $\mathcal{C}$ is a set $W \subseteq \bigsqcup_{k\geq 1}\mathcal{C}_k$ such that, for every k-morphism $(f\colon x \to y) \in W$, there exists a k-morphism $(g\colon y \to x) \in W$ and $(k+1)$-morphisms $(\alpha\colon 1_x \to gf) \in W$ and $(\beta\colon fg \to 1_y) \in W$. An n-morphism is *weakly invertible* if there exists a structure of reversibility containing it.

Remark 4.4.11. This definition has a 'coinductive' nature. In case that $\mathcal{C}$ is an n-category, we can equivalently proceed by downwards induction: an n-morphism is weakly invertible if and only if it is invertible and a k-morphism $f\colon x \to y$ for $k < n$ is weakly invertible if and only if there is a k-morphism $g\colon y \to x$ and weakly invertible $(k+1)$-morphisms $\alpha\colon 1_x \to gf$ and $\beta\colon fg \to 1_y$. For $n = 2$, this recovers the usual notion of an *equivalence* in a 2-category. A much more thorough discussion of equivalences internal to higher categories can be found in [OR24].

Definition 4.4.12 ([Lou21, Définition 5.2.1])**.** The unique functor $N^{\natural}_{\mathcal{O}}\colon \mathbf{Cat}^{\mathrm{st}}_{\omega} \to \mathbf{msSet}$ such that $UN^{\natural}_{\mathcal{O}} = N_{\mathcal{O}}$ and an n-simplex of $N_{\mathcal{O}}(\mathcal{C})$ is marked if and only if the corresponding functor $\mathcal{O}[n] \to \mathcal{C}$ sends the unique n-dimensional atom to a weakly invertible n-morphism in $\mathcal{C}$ is the *natural nerve*.

Remark 4.4.13. This naming convention is borrowed from [OR21, Construction 4.18].

For a strict ω-category $\mathcal{C}$, the marked simplicial set $N^{\natural}_{\mathcal{O}}(\mathcal{C})$ can be understood as a model for the induced 'weak ω-category', in the sense of the following Theorem.

Theorem 4.4.14 ([Lou21, Théorème 5.2.12])**.** *For $n \in \{0, 1, \dots, \infty\}$ and a strict (ω, n)-category $\mathcal{C}$, the natural nerve $N^{\natural}_{\mathcal{O}}(\mathcal{C})$ is an n-complicial set.* □

Definition 4.4.15. For $n \in \{0, 1, \dots, \infty\}$, the composite

$$\mathbf{Cat}^{\mathrm{st}}_{(\omega,n)} \xrightarrow{N^{\natural}_{\mathcal{O}}} \mathbf{msSet}_{(\infty,n)} \xrightarrow{R_n} \mathrm{Fun}\left(\Theta_n^{op}, \mathbf{sSet}\right)_{\mathrm{proj},(\infty,n)} \xrightarrow{\delta_n^*} \mathrm{Fun}\left((\Delta^{\times n})^{op}, \mathbf{sSet}\right)_{\mathrm{proj},(\infty,n)}$$

induces a functor $N^{\natural}_n\colon N\left(\mathbf{Cat}^{\mathrm{st}}_{(\omega,n)}\right) \to \mathbf{Cat}_{(\infty,n)}$, mapping a strict (ω, n)-category $\mathcal{C}$ to the n-fold complete Segal space

$$([k_1], \dots, [k_n]) \mapsto \mathrm{Map}\left(N^{RS}_{\mathcal{O}}\left(\delta_n\left([k_1], \dots, [k_n]\right)\right), N^{\natural}_{\mathcal{O}}(\mathcal{C})\right)^{\simeq}.$$

Proof. Theorem 4.4.14 implies that $N^{\natural}_{\mathcal{O}}$ takes value in the subcategory of fibrant objects. Theorems 4.2.6 and 4.3.18 imply that R_n and δ_n^* are right Quillen functors. □

Lastly, we establish the consistency of these two approaches.

Lemma 4.4.16. *For any gaunt ω-category $\mathcal{C}$, we have $N^{RS}_{\mathcal{O}}(\mathcal{C}) = N^{\natural}_{\mathcal{O}}(\mathcal{C})$.*

Proof. If $n \geq 0$ and $\mathcal{C}$ is a gaunt n-category, a downward induction shows that every weakly invertible morphism is an identity morphism. Thus, by definition, $N^{RS}_{\mathcal{O}}(\mathcal{C}) = N^{\natural}_{\mathcal{O}}(\mathcal{C})$. □

Proposition 4.4.17. *For any $n \geq 0$ and gaunt n-category $\mathcal{C}$, we have*

$$N^{\natural}_m(\mathcal{C}) = N_m(\mathcal{C}) = \varepsilon_n^{m-n} N_n(\mathcal{C}), \qquad m \geq n.$$

Proof. For any $m \geq n$ and $([k_1], \dots, [k_m]) \in (\Delta^{\times m})^{op}$, an l-simplex of

$$\mathrm{Map}\left(N^{RS}_{\mathcal{O}}\left(\delta_m\left([k_1], \dots, [k_m]\right)\right), N^{\natural}_{\mathcal{O}}(\mathcal{C})\right)^{\simeq}$$

corresponds to a marked simplicial map $\left(\Delta^l\right)^{\#} \times N^{RS}_{\mathcal{O}}\left(\delta_m\left([k_1], \dots, [k_m]\right)\right) \to N^{\natural}_{\mathcal{O}}(\mathcal{C}) = N^{RS}_{\mathcal{O}}(\mathcal{C})$ by adjunction and Lemma 4.4.16. Now, consider the entire inclusion $N^{RS}_{\mathcal{O}}([l]) \hookrightarrow \left(\Delta^l\right)^{\#}$, where $N^{RS}_{\mathcal{O}}([l])$

is Δ^l with all simplices in dimension ≥ 2 and only the degenerate simplices in dimension 1 marked. The marked simplicial map in question is the same as a marked simplicial map

$$N_{\mathcal{O}}^{RS}([l]) \times N_{\mathcal{O}}^{RS}(\delta_m([k_1],\dots,[k_m])) \to N_{\mathcal{O}}^{RS}(\mathcal{C}),$$

which additionally takes the product of any 1-simplex in $N_{\mathcal{O}}^{RS}([l])$ and degenerate 1-simplex in $N_{\mathcal{O}}^{RS}(\delta_m([k_1],\dots,[k_m]))$ to a degenerate 1-simplex in $N_{\mathcal{O}}(\mathcal{C})$. The functor $N_{\mathcal{O}}^{RS}$ preserves products by inspection, so this is equivalently a functor $[l] \times \delta_m([k_1],\dots,[k_m]) \to \mathcal{C}$, which additionally takes the product of any morphism of $[l]$ and identity morphism of $\delta_m([k_1],\dots,[k_m])$ to an identity morphism in $\mathcal{C}$, by Theorem 4.3.4. This implies the functor is constant on the first factor, whence naturally

$$\begin{aligned}\mathrm{Map}\left(N_{\mathcal{O}}^{RS}(\delta_m([k_1],\dots,[k_m])), N_{\mathcal{O}}^{\natural}(\mathcal{C})\right)^{\simeq} &\cong c\mathbf{Cat}_m^{\mathrm{st}}(\delta_m([k_1],\dots,[k_m]),\mathcal{C}) \\ &\cong c\mathbf{Cat}_n^{\mathrm{st}}(\delta_n([k_1],\dots,[k_n]),\mathcal{C}).\end{aligned}$$

The second identifications holds by Lemma 4.2.2, (iv), since $\mathcal{C}$ is an n-category. □

Warning. The $N_n^{\natural}$ are not fully faithful for $n \geq 2$. Furthermore, they do not satisfy a compatibility for different values of n as the N_n do by the above result.

Corollary 4.4.18. *The fully faithful functors $N_n\colon N(\mathrm{Gaunt}_n) \hookrightarrow \mathbf{Cat}_{(\infty,n)}$, $n \geq 0$ induce a fully faithful functor $N_\omega\colon N(\mathrm{Gaunt}_\omega) \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$. For a gaunt n-category $\mathcal{C}$, it is given by*

$$N_\infty^{\natural}(\mathcal{C}) = N_\omega(\mathcal{C}) = \varepsilon_{\leq n} N_n(\mathcal{C}).$$ □

4.5 The Mapping Space Formula

To understand the geometric realization $|-| : \mathbf{CatSp} \to \mathbf{Sp}$ of Corollary 3.4.16 that turns a categorical spectrum into a 'bordism spectrum', we need to understand the ordinary geometric realization $|-| : \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Spc}$. To this end, we first note the

Proposition 4.5.1. *For any ∞-category $\mathcal{C}$, there is a canonical equivalence*

$$|\mathcal{C}| \simeq \left|[n] \mapsto \mathrm{map}_{\mathbf{Cat}_\infty}(N([n]),\mathcal{C})\right|.$$

Proof. This is a consequence of Theorem 3.2.4 and the uniqueness of adjoints. □

This formula is 'model-independent' as it only involves mapping spaces in $\mathbf{Cat}_\infty$ and the nerve functor $N(-)\colon N(\mathrm{Gaunt}_1) \hookrightarrow \mathbf{Cat}_\infty$. The rest of this Subsection is dedicated to proving the following generalization:

Theorem 4.5.2 (Mapping Space Formula)**.** *For any (∞,∞)-category $\mathcal{C}$, there is a canonical equivalence*

$$|\mathcal{C}| \simeq \left| [n] \mapsto \mathrm{map}_{\mathbf{Cat}_{(\infty,\infty)}} \left(N_\omega \left(\mathcal{O}[n] \right), \mathcal{C} \right) \right| .$$

Remark 4.5.3. It appears to be quite difficult to intuit this formula. Examining the proof, the equivalence can be roughly described as follows: For every $n \geq 0$, there is a composite

$$\mathrm{map}_{\mathbf{Cat}_{(\infty,\infty)}} \left(N_\omega(\mathcal{O}[n]), \mathcal{C} \right) \longrightarrow \mathrm{map}_{\mathbf{Cat}_{(\infty,\infty)}} \left(*, \mathcal{C} \right) \simeq \mathcal{C}^{\simeq} \longrightarrow |\mathcal{C}| ,$$

induced by restricting along the the 0-th vertex $* \hookrightarrow N_\omega \left(\mathcal{O}[n] \right)$. This is immediately compatible with the maps $[n] \to [m]$ in Δ that preserve the 0-th vertex. For a map $[n] \to [m]$ in Δ that does not preserve the 0-th vertex, there is nonetheless a canonical natural transformation in $\mathcal{C}$ given objectwise by the respective edge of the diagram $N_\omega \left(\mathcal{O}[n] \right) \to \mathcal{C}$ and its image in $|\mathcal{C}|$ becomes a natural equivalence. Together, these compatible maps induce the proposed equivalence.

Remark 4.5.4. Though we have not established a definition of a 'model of (∞,∞)-categories', any such model should come with an embedding $N_\omega \colon N(\mathrm{Gaunt}_\omega) \hookrightarrow \mathbf{Cat}_{(\infty,\infty)}$ as we have constructed in Corollary 4.4.18, and this formula can be interpreted in any such model.

The complicial model allows us to clarify the role of the orientals.

Theorem 4.5.5 ([Mae23, Theorem 2.1],[GOR23, Corollary 5.4])**.** *The monomorphism*

$$(\Delta^n)^\flat \hookrightarrow N^{RS}_{\mathcal{O}}(\mathcal{O}[n])$$

corresponding to $\mathrm{id}_{\mathcal{O}[n]}$ *is a complicial (inner) anodyne extension, in particular a weak equivalence in* $\mathbf{msSet}_{(\infty,n)}$ *for any* $n \in \{0, 1, \ldots, \infty\}$. □

Remark 4.5.6. This Theorem says that the n-th oriental is a fibrant replacement of the (minimally marked) n-simplex, i.e. a functor $\mathcal{O}[n] \to \mathcal{C}$ into an (∞,∞)-category $\mathcal{C}$ is essentially the same thing as a 'Δ^n-shaped' diagram in $\mathcal{C}$.

Proof of Theorem 4.5. There is a commutative triangle of ∞-categories

$$\begin{array}{ccc} N\left(\mathrm{Gaunt}_\omega\right) & & \\ \Big\downarrow{\scriptstyle N^{RS}_{\mathcal{O}}} & \searrow^{N_\omega} & \\ N\left(\mathbf{msSet}^{\circ}_{(\infty,\infty)}\right) & \xrightarrow{\sim} & \mathbf{Cat}_{(\infty,\infty)}. \end{array}$$

The equivalence is induced by the right Quillen equivalences of Theorems 4.2.8 and 4.3.20, the vertical map is well-defined by Theorem 4.4.14 and Lemma 4.4.16 and the commutativity is due to Corollary

4.4.18. Thus, it suffices to provide for any ∞-complicial set X a canonical equivalence

$$|X| \simeq \left| [n] \mapsto \mathrm{map}_{N\left(\mathbf{msSet}^{\circ}_{(\infty,\infty)}\right)} \left(N^{RS}_{\mathcal{O}} \left(\mathcal{O}[n] \right), X \right) \right|.$$

There are natural equivalences

$$\mathrm{map}_{N\left(\mathbf{msSet}^{\circ}_{(\infty,\infty)}\right)} \left(N^{RS}_{\mathcal{O}} \left(\mathcal{O}[n] \right), X \right) \simeq \mathrm{Map} \left(N^{RS}_{\mathcal{O}} \left(\mathcal{O}[n] \right), X \right)^{\simeq} \simeq \mathrm{Map} \left((\Delta^n)^{\flat}, X \right)^{\simeq}, \qquad \forall n \geq 0.$$

Indeed, $\mathbf{msSet}_{(\infty,\infty)}$ is a simplicial model category by Corollary 4.3.16, so the first equivalence is the standard identification of mapping spaces in the underlying ∞-category and the second equivalence is induced by mapping the weak equivalence (of cofibrant objects) $(\Delta^n)^{\flat} \hookrightarrow N^{RS}_{\mathcal{O}} \left(\mathcal{O}[n] \right)$ of Theorem 4.5.5 into the fibrant object X. Thus, it suffices to establish a canonical equivalence

$$|X| \simeq \left| [n] \mapsto \mathrm{Map} \left((\Delta^n)^{\flat}, X \right)^{\simeq} \right|.$$

The right Quillen equivalence $\mathrm{Sing} \circ i$ of Corollary 4.3.26 induces a fully faithful functor

$$\begin{aligned} N\left(\mathbf{msSet}^{\circ}_{(\infty,\infty)}\right) &\xrightarrow{\sim} N\left(\mathrm{Fun}\left(t\Delta^{op}, \mathbf{sSet}\right)^{\circ}_{\mathrm{inj},(\infty,n)}\right) \\ &\hookrightarrow N\left(\mathrm{Fun}\left(t\Delta^{op}, \mathbf{sSet}\right)^{\circ}_{\mathrm{inj}}\right) \\ &\xrightarrow{\sim} \mathrm{Fun}\left(N\left(t\Delta\right)^{op}, \mathbf{Spc}\right), \end{aligned}$$

using [Lur09b, Proposition 4.2.4.4] once more. The simplicial functor $(-)^{\#} \colon \mathbf{sSet} \to \mathbf{msSet}$ preserves fibrant-cofibrant objects by Remark 4.3.11, so we obtain a commutative triangle

$$\begin{array}{ccc} \mathbf{Spc} & & \\ \downarrow{\scriptstyle (-)^{\#}} & \searrow^{\delta} & \\ N\left(\mathbf{msSet}^{\circ}_{(\infty,\infty)}\right) & \hookrightarrow & \mathrm{Fun}\left(N\left(t\Delta\right)^{op}, \mathbf{Spc}\right). \end{array}$$

Indeed, the down-right composite takes a Kan complex X to the functor

$$[n] \mapsto \mathrm{Map}\left((\Delta^n)^{\flat}, X^{\#}\right)^{\simeq} = \mathrm{Map}\left(\Delta^n, X\right), \qquad [n]_t \mapsto \mathrm{Map}\left((\Delta^n)^{\#}, X^{\#}\right)^{\simeq} = \mathrm{Map}\left(\Delta^n, X\right),$$

which is essentially constant since the standard simplices are weakly contractible. Thus, the adjunction $\mathrm{colim}_{N(t\Delta)^{op}} \colon \mathrm{Fun}\left(N\left(t\Delta\right)^{op}, \mathbf{Spc}\right) \rightleftarrows \mathbf{Spc} \colon \delta$ restricts to an adjunction

$$\operatorname*{colim}_{N(t\Delta)^{op}} \colon N\left(\mathbf{msSet}^{\circ}_{(\infty,\infty)}\right) \rightleftarrows \mathbf{Spc} \colon (-)^{\#}.$$

The geometric realization is definitionally given by this left adjoint, so the canonical equivalence $|X| \simeq \left|[n] \mapsto \mathrm{Map}\left(\left(\Delta^n\right)^\flat, X\right)^{\simeq}\right|$ follows by applying the next Lemma. □

Lemma 4.5.7. *The inclusion $N(\Delta)^{op} \hookrightarrow N(t\Delta)^{op}$ is cofinal.*

Proof. For $n \geq 0$, the ∞-category $N(\Delta)^{op} \times_{N(t\Delta)^{op}} N(t\Delta)^{op}_{[n]/} = N(\Delta)^{op}_{[n]/}$ is weakly contractible since it has an initial object. For $n \geq 1$, the ∞-category $N(\Delta)^{op} \times_{N(t\Delta)^{op}} N(t\Delta)^{op}_{[n]_t/} = N(\Delta)^{op}_{[n]/}$ (indeed, every morphism $[m] \to [n]_t$ factors uniquely through the map $[n] \to [n]_t$) is weakly contractible since it has an initial object. Thus, the claim follows from [Lur09b, Theorem 4.1.3.1]. □

5 Discussion

In this Section, having established the theory of categorical spectra and the mapping space formula, we discuss and interpret the first non-trivial example as well as give a future outlook.

5.1 The (∞, ∞)-category of Bordisms

The (∞, n)-category of bordisms $\mathbf{Bord}_n$ was properly defined in [CS19] for all $n \geq 0$. Though the '(∞, ∞)-category of bordisms $\mathbf{Bord}_\infty$' has been the subject of discussion, it has not properly appeared in the literature (a complicial model was proposed in [Ver17], yet never published). It is the goal of this Subsection to propose a definition that fits into our framework.

First of all, we need to compare the formalism of [CS19] to ours. To this end, let $\mathbf{Seg}_{(\infty,n)}$ denote the full subcategory of $\mathrm{Fun}\left(N\left(\Delta^{\times n}\right)^{op}, \mathbf{Spc}\right)$ on the *n-fold Segal spaces* of Remark 3.2.6. We implement the 'completion' of [CS19, Definition 2.14].

Proposition 5.1.1. *The inclusion $\mathbf{Cat}_{(\infty,n)} \subseteq \mathbf{Seg}_{(\infty,n)}$ admits a relatively left exact left adjoint $L\colon \mathbf{Seg}_{(\infty,n)} \to \mathbf{Cat}_{(\infty,n)}$ with repect to the essential image of $\varepsilon_0^n\colon \mathbf{Spc} \hookrightarrow \mathbf{Cat}_{(\infty,n)} \subseteq \mathbf{Seg}_{(\infty,n)}$.*

Proof. This is proven inductively, the base case being trivial. In the inductive step, consider the left adjoint to the inclusion

$$\mathrm{Fun}\left(N(\Delta)^{op}, \mathbf{Seg}_{(\infty,n)}\right) \xrightarrow{L_*} \mathrm{Fun}\left(N(\Delta)^{op}, \mathbf{Cat}_{(\infty,n)}\right),$$

given by post-composing with $L\colon \mathbf{Seg}_{(\infty,n)} \to \mathbf{Cat}_{(\infty,n)}$, which is left exact relative to spaces. Thus, it preserves the simplicial objects satisfying the Segal condition and where X_0 is a constant n-fold Segal space, i.e. it restricts to a left adjoint $L_*\colon \mathbf{Seg}_{(\infty,n+1)} \to SS_{\mathbf{Cat}_{(\infty,n)}}$ to the inclusion, which is again left exact relative to spaces as limits in functor categories can be computed pointwise. The composition with the left adjoint $SS_{\mathbf{Cat}_{(\infty,n)}} \to CSS_{\mathbf{Cat}_{(\infty,n)}}$ to the inclusion of Theorem 3.1.28, which is left exact relative to spaces by Corollary 3.1.34, completes the induction. □

Remark 5.1.2. From Proposition 4.4.7, recall that a strict n-category $\mathcal{C}$ induces a (discrete) n-fold Segal space. It is reasonable to suspect that its completion is canonically equivalent to the (∞, n)-category $N_n^{\natural}(\mathcal{C})$ of Definition 4.4.15. This is not hard to see for $n = 1$, but the general case is open.

For a pointed n-fold Segal space $* \to X$, we define as in Proposition 3.1.42 the $(n-1)$-fold Segal space $\Omega X \simeq X_{1,\bullet,\dots,\bullet} \times_{X_{0,\bullet,\dots,\bullet}^{\times 2}} \{*\}$. An n-fold Segal space X is *connected* if $X_{0,\bullet,\dots,\bullet} \simeq *$ ([CS19, Definition 3.13]). If X is connected, then $\Omega X \simeq X_{1,\bullet,\dots,\bullet}$. Furthermore, an n-fold Segal space X is *k-connected*, $0 \leq k \leq n$ if $\Omega^l X$ is connected for $l = 0, \dots, k$.

Corollary 5.1.3. *For any $n \geq 1$, there is a canonical commutative square*

$$\begin{array}{ccc} \mathbf{Seg}_{(\infty,n)} & \xrightarrow{L} & \mathbf{Cat}_{(\infty,n),*} \\ \downarrow{\scriptstyle \Omega} & & \downarrow{\scriptstyle \Omega} \\ \mathbf{Seg}_{(\infty,n-1),*} & \xrightarrow{L} & \mathbf{Cat}_{(\infty,n-1),*}. \end{array}$$

Proof. This follows from the construction in Proposition 5.1.1. Indeed, $\Omega L_* \simeq L\Omega$ since L preserves the pullbacks over spaces that define Ω, and the unit map of $SS_{\mathbf{Cat}_{(\infty,n)}} \rightrightarrows CSS_{\mathbf{Cat}_{(\infty,n)}}$ is inverted by Ω as proven in Theorem 3.1.44. □

Remark 5.1.4. This corrects [CS19, Remark 7.8], where these are claimed to not necessarily commute.

Corollary 5.1.5. *For any $n \geq 0$, $0 \leq k \leq n$, the completion of a k-connected n-fold Segal space is a $(k+1)$-connective (∞, n)-category.*

Proof. A connected n-fold Segal space X is a monoid object in $(n-1)$-fold Segal spaces, so that L_*X is a monoid object in $(\infty, n-1)$-categories and Theorem 3.1.44 implies that LX is a connective (∞, n)-category. If X is k-connected for some $0 \leq k \leq n$, then $\Omega^l X$ is connected for all $l = 0, \dots, k$, hence $L\Omega^l X \simeq \Omega^l LX$ is connective for all $l = 0, \dots, k$ and thus LX is $(k+1)$-connective. □

Definition 5.1.6 ([CS19, Section 4.3])**.** The simplicial space Int is given by $[k] \mapsto \mathrm{Int}_k$, where Int_k is the space of smooth simplices in the smooth manifold

$$\mathfrak{Int}_k := \left\{ (a_0, \dots, a_k, b_0, \dots, b_k) \in \mathbb{R}^{2(k+1)} \;\middle|\; \begin{array}{l} a_i < b_i \text{ for } 0 \leq i \leq k, \\ a_{i-1} \leq a_i \text{ and } b_{i-1} \leq b_i \text{ for } 1 \leq i \leq k \end{array} \right\},$$

i.e. an m-simplex of Int_k is given by smooth maps $a_i, b_i \colon |\Delta^m|_e \to \mathbb{R}$, $i = 0, \dots, k$, where $|\Delta^m|_e = \{(x_0, \dots, x_m) \in \mathbb{R}^{m+1} | x_0 + \dots + x_m = 1\}$ is the 'extended m-simplex', $a_i(s) < b_i(s)$ for $i = 0, \dots, k$ and $a_{i-1}(s) \leq a_i(s)$, $b_{i-1}(s) \leq b_i(s)$ for $i = 1, \dots, k$ and $s \in |\Delta^m|_e$.

Notation 5.1.7. An m-simplex of Int_k is also denoted as $I = (I_0(s) \leq \ldots \leq I_k(s))_{s \in |\Delta^m|_e}$, where $I_i(s) = [a_i(s), b_i(s)] \cap (a_0(s), b_k(s))$ is interpreted as a closed interval in $(a_0(s), b_k(s))$ for any $s \in |\Delta^m|_e$.

For $k_1, \ldots, k_d \geq 0$ and an m-simplex $I = (I^i)_{1 \leq i \leq d} = \left((I_0^i(s) \leq \ldots \leq I_{k_i}^i(s))\right)_{1 \leq i \leq d,\, s \in |\Delta^m|_e}$ of $\mathrm{Int}_{k_1} \times \ldots \times \mathrm{Int}_{k_d}$, the 'boxing' of I is the subspace

$$B(I) \coloneqq \bigcup_{s \in |\Delta^m|_e} \left(a_0^1(s), b_{k_1}^1(s)\right) \times \ldots \times \left(a_0^d(s), b_{k_d}^d(s)\right) \times \{s\} \subseteq \mathbb{R}^d \times |\Delta^m|_e \,.$$

Definition 5.1.8. [CS19, Definition 6.1] Let $n \geq 0$ and $l \geq -n$ be integers, $d = n + l$. For $k_1, \ldots, k_d \geq 0$, let $\left(\mathbf{PBord}_n^l\right)_{k_1, \ldots, k_d}$ denote the simplicial set whose m-simplices, $m \geq 0$ are given by tuples $\left(M, I = \left(I_0^i(s) \leq \ldots \leq I_{k_i}^i(s)\right)_{1 \leq i \leq d,\, s \in |\Delta^m|_e}\right)$, where

(1) For $1 \leq i \leq d$, $\left(I_0^i(s) \leq \ldots \leq I_{k_i}^i(s)\right)_{s \in |\Delta^m|_e}$ is an m-simplex of Int_{k_i}.

(2) M is a closed and bounded $(n+m)$-dimensional smooth submanifold of $\mathbb{R}^\infty \times B(I)$, such that the composite $\pi \colon M \subseteq \mathbb{R}^\infty \times B(I) \to B(I)$ is a proper map.

For $S \subseteq \{1, \ldots, d\}$, let $p_S \colon M \xrightarrow{\pi} B(I) \to \mathbb{R}^S \times |\Delta^m|_e$ denote the composite projection.

(3) The map $p_\emptyset \colon M \to |\Delta^m|_e$ is a submersion that is trivial outside $|\Delta^m| \subseteq |\Delta^m|_e$.

(4) For every $1 \leq i \leq d$ and $0 \leq j_i \leq k_i$, $p_{\{i, \ldots, d\}}$ is submersive on $p_{\{i\}}^{-1}\left(\bigcup_{s \in |\Delta^m|_e} I_{j_i}^i(s) \times \{s\}\right)$.

This defines an d-fold simplicial space[4] $\mathbf{PBord}_n^l$.

Remark 5.1.9. In [CS19, Definition 6.1], this definition is stated relative to a finite-dimensional $\mathbb{R}$-vector space V instead of $\mathbb{R}^\infty$. Then, they take a homotopy colimit over the finite-dimensional subspaces $V \leq \mathbb{R}^\infty$. However, this homotopy colimit can be computed in $\mathrm{Fun}\left(N\left(\Delta^{\times d}\right)^{op}, \mathbf{Spc}\right)$, i.e. pointwise, and filtered colimits in $\mathbf{Spc}$ are computed strictly, hence this is a simplification.

Remark 5.1.10. Intuitively, this (∞, n)-category should be understood as having k-bordisms as k-morphisms for $k \leq n$ and higher morphisms given by diffeomorphisms, though in reality all bordisms are n-dimensional and instead 'thickened up' with various collars along the coordinate axes. The intuition is explained in [CS19, Remark 5.3], a precise result is [CS19, Proposition 8.17].

Proposition 5.1.11 ([CS19, Proposition 6.2, Remark 6.5])**.** *For any $n \geq 0$, $l \geq -n$, $\mathbf{PBord}_n^l$ is an $(l-1)$-connected*[5] *d-fold Segal space, canonically pointed by $(\emptyset, ((0,1), \ldots, (0,1)))$.* □

Definition 5.1.12. For any $n \geq 0$, $l \geq -n$, the (∞, d)-category $\mathbf{Bord}_n^l$ is the completion $L\mathbf{PBord}_n^l$.

[4] This being a space is a consequence of an analogue of [CS19, Lemma 5.7].

[5] This condition is vacuous if $l \leq 0$.

Proposition 5.1.13. *For any $n \geq 0$, $l \geq -n$ and $d \geq k \geq 0$, there is a canonical equivalence of pointed $(\infty, d-k)$-categories $\Omega^k \mathbf{Bord}_n^l \xrightarrow{\sim} \mathbf{Bord}_n^{l-k}$.*

Proof. This is immediate by combining [CS19, Theorem 7.4] and Corollary 5.1.3. □

Remark 5.1.14. In particular, combining this with [CS19, Proposition 8.20], there is a canonical equivalence of symmetric monoidal (ordinary) categories $\mathbf{nCob} \xrightarrow{\sim} h\left(\Omega^{n-1}\mathbf{Bord}_n^0\right)$ (compare [CS19, Remark 10.2]). Here, $\mathbf{nCob}$ is the category of n-dimensional bordisms ([CS19, Definition 8.18]).

For $n \geq 0$, $l \geq -n$, there is a canonical map $\theta'_{n,l} \colon \mathbf{PBord}_n^l \to \kappa_{n+l}\mathbf{PBord}_{n+1}^l$ of $(n+l)$-fold simplicial spaces (here, we extend the formulae of Remark 3.2.9 to n-fold Segal spaces) given by (compare to [CS19, Section 10.2])

$$\left(M, I = \left(I_0^i(s) \leq \ldots \leq I_{k_i}^i(s)\right)_{1 \leq i \leq d,\, s \in |\Delta^m|_e}\right) \mapsto \left(M \times (0,1), \left(I, ((0,1))_{s \in |\Delta^m|_e}\right)\right).$$

This is adjoint to a map $\varepsilon_{n+l}\mathbf{PBord}_n^l \to \mathbf{PBord}_{n+1}^l$, which induces a map

$$\varepsilon_{n+l}\mathbf{Bord}_n^l \simeq L\varepsilon_{n+l}\mathbf{PBord}_n^l \to \mathbf{Bord}_{n+1}^l,$$

where the canonical equivalence $\varepsilon_{n+l}L \simeq L\varepsilon_{n+1}$ follows from the uniqueness of adjoints. This ultimately induces a canonical map $\theta_{n,l} \colon \varepsilon_{\leq n+l}\mathbf{Bord}_n^l \to \varepsilon_{\leq n+l+1}\mathbf{Bord}_{n+1}^l$.

Definition 5.1.15. The categorical *bordism spectrum (of smooth manifolds)* $\mathbf{Bord}_\infty$ is the connective categorical spectrum given as the filtered colimit $\left(\mathbf{Bord}_\infty^l\right)_{l \geq 0} = \mathrm{colim}_{n \geq 0}\left(\varepsilon_{\leq n+l}\mathbf{Bord}_n^l\right)_{l \geq 0}$.

Proof. For any $n \geq 0$, the categorical spectrum $\left(\varepsilon_{\leq n+l}\mathbf{Bord}_n^l\right)_{l \geq 0}$ is given by the canonical equivalences of Proposition 5.1.13. For $l > 0$, the (∞,∞)-category $\varepsilon_{\leq n+l}\mathbf{Bord}_n^l$ is l-connective by Proposition 5.1.11 and Corollary 5.1.5, so this categorical spectrum is connective. The diagram

$$\begin{array}{ccccc}
 & & \Omega\mathbf{PBord}_n^l & \xrightarrow{\simeq} & \mathbf{PBord}_n^{l-1} \\
 & \overset{\Omega\theta'_{n,l}}{\swarrow} & \downarrow & & \downarrow{\scriptstyle \theta'_{n,l-1}} \\
\Omega\kappa_{n+l}\mathbf{PBord}_{n+1}^l & \simeq & \kappa_{n+l-1}\Omega\mathbf{PBord}_{n+1}^l & \xrightarrow{\simeq} & \kappa_{n+l-1}\mathbf{PBord}_{n+1}^{l-1}
\end{array}$$

commutes by inspection of the map constructed in [CS19, Theorem 7.4][6]. This implies that the maps $\theta_{n,l}$, $l \geq 0$ naturally assemble into a map $\theta_n \colon \left(\varepsilon_{\leq n+l}\mathbf{Bord}_n^l\right)_{l \geq 0} \to \left(\varepsilon_{\leq n+l+1}\mathbf{Bord}_{n+1}^l\right)_{l \geq 0}$ of connective categorical spectra. These are the transfer maps in the diagram whose filtered colimit defines $\mathbf{Bord}_\infty$. Lastly, the full subcategory of connective categorical spectra is stable under small colimits (e.g. by the Recognition Principle as B^∞ is a left adjoint) so that $\mathbf{Bord}_\infty$ is connective. □

[6] The loop object can be explicitly modeled by evaluating the first index at [1], since these Segal spaces are connected, and in that case the diagram commutes strictly.

Remark 5.1.16. Following [CS19, Section 9], it is also possible to define variants like $\mathbf{Bord}_\infty^{\mathrm{or}}$ for oriented bordism of smooth manifolds, or other tangential structures.

This connective categorical spectrum corresponds to a symmetric monoidal (∞,∞)-category, the (∞,∞)-*category of bordisms*, whose underlying (∞,∞)-category is $\mathbf{Bord}_\infty^0$. Intuitively, this (∞,∞)-category has n-bordisms as n-morphisms for all $n \geq 0$ and the symmetric monoidal structure is given by disjoint unions. More generally, $\mathbf{Bord}_\infty^l$ is an l-connective (∞,∞)-category that has n-bordisms as $(n+l)$-morphisms for all $n \geq 0$. The monoid $\pi_0 \left|\mathbf{Bord}_\infty^0\right| \cong \{[\emptyset],[*]\}$ under disjoint union is a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so $\mathbf{Bord}_\infty^0$ is weakly grouplike. Thus, the 'bordism spectrum' $|\mathbf{Bord}_\infty|$ is given by the spaces $\left|\mathbf{Bord}_\infty^l\right|$, $l \geq 0$ by Corollary 3.4.30. The mapping space formula gives

$$\left|\mathbf{Bord}_\infty^l\right| \simeq \left|[n] \mapsto \mathrm{map}_{\mathbf{Cat}_{(\infty,\infty)}}\left(N_\omega\left(\mathcal{O}[n]\right), \mathbf{Bord}_\infty^l\right)\right|.$$

Intuitively, a functor $N_\omega\left(\mathcal{O}[n]\right) \to \mathbf{Bord}_\infty^l$ is a Δ^n-shaped diagram in $\mathbf{Bord}_\infty^l$ (Remark 4.5.6), i.e. one associates $(m-l)$-dimensional manifolds to the m-dimensional faces of Δ^n that satisfy bordism relations according to how these faces intersect. This morally recovers (a smooth version of) what is called an $(n-l)$-dimensional manifold $[n]$-ad in [Qui95, Section 3.1] or a Δ^n-ad of degree l in [LM14, Section 6]. Thus, our construction of $|\mathbf{Bord}_\infty|$ crystallizes as a higher categorical generalization of the construction of Thom spectra in [Qui95, Sections 3.4, 6.1] and [LM14, Definition 15.8, Appendix B] respectively. This justifies the claimed lineage of our philosophy.

5.2 Outlook

The technical framework for our philosophy has thus been developed, yet much still remains open. We list some future research topics:

- It should be possible to give a direct description of the cohomology theory $H^\bullet(-;|\mathcal{C}|)$ for a categorical spectrum $\mathcal{C}$, analogous to the one in [LM14, Section 14].
- For a categorical spectrum $\mathcal{C}$, when is the spectrum $|\mathcal{C}|$ obtained as the levelwise geometric realization of $\mathcal{C}$? The connective case was addressed in Corollary 3.4.30. To address the not necessarily connective case, we conjecture that this holds if $\mathcal{C}$ 'has duals' in an appropriate sense generalizing [Lur09c, Definition 2.3.16].
- There should be a 'categorical spectrum of algebraic bordisms' $\mathbf{Bord}_\infty^{\mathrm{alg}}$, analogs to $\mathbf{Bord}_\infty$, where compact n-manifolds are replaced by n-dimensional (symmetric) Poincaré duality complexes. These are, roughly speaking, chain complexes $C_\bullet$ equipped with a particular chain equivalence $C^{n-\bullet} \xrightarrow{\sim} C_\bullet$ ('generalized Poincaré duality'). These also make sense for negative n, so $\mathbf{Bord}_\infty^{\mathrm{alg}}$ will be highly non-connective.

We expect that it is possible to construct $\mathbf{Bord}_\infty^{\mathrm{alg}}$ as in Section 5.1 and identify $\left|\mathbf{Bord}_\infty^{\mathrm{alg}}\right| \simeq L^s(\mathbb{Z})$ as a (symmetric) L-theory spectrum by building on [CDH+20, Sections 2.3, 3.4, 4.4]. Furthermore, taking singular chain complexes should induce a map $\mathbf{Bord}_\infty^{\mathrm{or}} \to \mathbf{Bord}_\infty^{\mathrm{alg}}$ of categorical spectra, whose geometric realization implements the *Ranicki-Sullivan orientation* $MSO \to L^s(\mathbb{Z})$.

A Cartesian Closed ∞-categories and Localization

In this Appendix, we introduce cartesian closed ∞-categories and their full subcategories compatible with the cartesian closed structure, the exponential ideals.

Definition A.0.1. An ∞-category $\mathcal{C}$ is *cartesian* if it admits finite products. A functor $f\colon \mathcal{C} \to \mathcal{D}$ of cartesian ∞-categories is *cartesian* if it preserves finite products. A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ of a cartesian ∞-category is *cartesian* if it stable under equivalences and finite products. In particular, $\mathcal{D}$ is a cartesian ∞-category and the inclusion $\mathcal{D} \subseteq \mathcal{C}$ is a cartesian functor.

Definition A.0.2. An ∞-category $\mathcal{C}$ is *cartesian closed* if it is cartesian and, for any object $x \in \mathcal{C}$, the product functor $(-) \times x\colon \mathcal{C} \to \mathcal{C}$ admits a right adjoint, denoted $[x,-] \coloneqq [x,-]_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. A cartesian functor $f\colon \mathcal{C} \to \mathcal{D}$ of cartesian closed ∞-categories is *cartesian closed* if the natural transformation $f\,[x,-]_{\mathcal{C}} \to [fx, f(-)]_{\mathcal{D}}$, induced by $f\,[x,-]_{\mathcal{C}} \times fx \simeq f([x,-]_{\mathcal{C}} \times x) \to f(-)$ under the adjunction, is a natural equivalence for any object $x \in \mathcal{C}$. A cartesian subcategory $\mathcal{D} \subseteq \mathcal{C}$ of a cartesian closed ∞-category is an *exponential ideal* if, for any objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we have $[x,y]_{\mathcal{C}} \in \mathcal{D}$.

Proposition A.0.3. *If $\mathcal{C}$ is a cartesian closed ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ an exponential ideal, then $\mathcal{D}$ is a cartesian closed ∞-category and the inclusion $\mathcal{D} \subseteq \mathcal{C}$ is cartesian closed.*

Proof. The definition implies that the adjunction $(-) \times x\colon \mathcal{C} \rightleftarrows \mathcal{C}\colon [x,-]_{\mathcal{C}}$ for any object $x \in \mathcal{D}$ restricts to an adjunction $(-) \times x\colon \mathcal{D} \rightleftarrows \mathcal{D}\colon [x,-]_{\mathcal{C}}\,|_{\mathcal{D}}$. □

Example A.0.4. Let $\mathcal{X}$ be an ∞-topos. For any object $x \in \mathcal{X}$, the functor $(-) \times x\colon \mathcal{X} \to \mathcal{X}$ preserves small colimits since they are universal in $\mathcal{X}$. Thus, by the adjoint functor theorem, $(-) \times x$ admits a right adjoint, i.e. $\mathcal{X}$ is a cartesian closed ∞-category.

Example A.0.5. The ∞-category $\mathbf{Cat}_\infty$ is cartesian closed. For an ∞-category $\mathcal{C}$, the mapping space functor $\mathrm{Fun}(\mathcal{C},-)\colon \mathbf{Cat}_\infty \to \mathbf{Cat}_\infty$ is right adjoint to the functor $(-) \times \mathcal{C}$ (this adjunction is induced by an $\mathbf{sSet}$-enriched adjunction as in [Lur09b, Corollary 5.2.4.5]). Similarly, the ∞-category $\mathbf{Spc}$ is cartesian closed.

Proposition A.0.6. *Let $\mathcal{C}$ be a cartesian closed ∞-category. There is a functor $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ that preserves all limits that exist in $\mathcal{C}$ in each variable separately and such that, for any object $x \in \mathcal{C}$, $[x,-]$ is right adjoint to $(-) \times x$.*

Proof. The product functor $(-)\times(-)\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ curries to a functor $\mathcal{C}\to\mathrm{Fun}(\mathcal{C},\mathcal{C})$, which factors through the full subcategory $\mathrm{Fun}^L(\mathcal{C},\mathcal{C})$ on the left adjoint functors since $\mathcal{C}$ is cartesian closed. This subcategory is canonically equivalent to $\mathrm{Fun}^R(\mathcal{C},\mathcal{C})^{op}$ by passing to the right adjoints ([Lur09b, Proposition 5.2.6.2]), so we obtain a composite $\mathcal{C}\to\mathrm{Fun}^R(\mathcal{C},\mathcal{C})^{op}$. Taking opposites and un-currying yields the desired functor $[-,-]:\mathcal{C}^{op}\times\mathcal{C}\to\mathcal{C}$.

The functor $\mathcal{C}\to\mathrm{Fun}(\mathcal{C},\mathcal{C})$, $x\mapsto(-)\times x$ preserves all colimits that exist in $\mathcal{C}$ since these are computed pointwise ([Lur09b, Corollary 5.1.2.3]) and taking the product with an object is a left adjoint functor, hence preserves all colimits that exist. This implies that $\mathcal{C}\to\mathrm{Fun}^L(\mathcal{C},\mathcal{C})$ also preserves all colimits that exist in $\mathcal{C}$, hence $\mathcal{C}^{op}\to\mathrm{Fun}^R(\mathcal{C},\mathcal{C})$ preserves all limits that exist in $\mathcal{C}$ and the un-curried functor $[-,-]$ preserves all limits that exist in $\mathcal{C}$ in the first variable. □

Proposition A.0.7. *Let $\mathcal{C}$ be a cartesian closed ∞-category and $\mathcal{D}\subseteq\mathcal{C}$ a coreflective, cartesian subcategory. Then, $\mathcal{D}$ is a cartesian closed ∞-category. Furthermore, if $\mathcal{E}\subseteq\mathcal{D}$ is an exponential ideal in $\mathcal{C}$, then it is an exponential ideal in $\mathcal{D}$.*

Proof. The assumption implies that the inclusion $\mathcal{D}\subseteq\mathcal{C}$ is cartesian. For any object $x\in\mathcal{D}$, we obtain a commutative square of ∞-categories

$$\begin{array}{ccc}\mathcal{D} & \xrightarrow{(-)\times x} & \mathcal{D}\\ \cap\!| & & \cap\!| \\ \mathcal{C} & \xrightarrow{(-)\times x} & \mathcal{C}.\end{array}$$

The composite along the left and bottom edge is left adjoint to the composite

$$\mathcal{C}\xrightarrow{[x,-]_{\mathcal{C}}}\mathcal{C}\xrightarrow{R}\mathcal{D}.$$

Thus, we obtain a restricted adjunction $(-)\times x\colon\mathcal{D}\rightleftarrows\mathcal{D}\colon R\,[x,-]_{\mathcal{C}}\,|_{\mathcal{D}}$. This implies that $\mathcal{D}$ is cartesian closed. If $\mathcal{E}\subseteq\mathcal{D}$ is an exponential ideal in $\mathcal{C}$, then $[x,y]_{\mathcal{D}}\simeq R[x,y]_{\mathcal{C}}\simeq[x,y]_{\mathcal{C}}\in\mathcal{E}$ for any objects $x\in\mathcal{D}$ and $y\in\mathcal{E}$ since $R|_{\mathcal{D}}\simeq\mathrm{id}_{\mathcal{D}}$. Thus, $\mathcal{E}$ is an exponential ideal in $\mathcal{D}$. □

Remark A.0.8. In this case, the inclusion $\mathcal{D}\subseteq\mathcal{C}$ need *not* be a cartesian closed functor.

Proposition A.0.9. *Let $\mathcal{C}$ be a cartesian closed ∞-category, $\mathcal{D}\subseteq\mathcal{C}$ a reflective subcategory and S the strongly saturated class of morphisms inverted by the left adjoint $L\colon\mathcal{C}\to\mathcal{D}$. The, $\mathcal{D}\subseteq\mathcal{C}$ is a cartesian subcategory and the following conditions are equivalent:*

(1) $\mathcal{D}$ is an exponential ideal in $\mathcal{C}$,

(2) S is closed under products,

(3) L preserves finite products.

Proof. As a reflective subcategory, $\mathcal{D}$ is stable under all limits that exist in $\mathcal{C}$, so it is clearly cartesian. In particular, $\mathcal{D}$ contains the terminal object, so L preserves it, and (3) $\Leftrightarrow$ L preserves binary products.

Let objects $x \in \mathcal{C}$, $y \in \mathcal{D}$ and a morphism $f\colon s \to t$ in S be given. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{C}}\left(t, [x,y]_{\mathcal{C}}\right) & \xrightarrow{f^*} & \mathrm{map}_{\mathcal{C}}\left(s, [x,y]_{\mathcal{C}}\right) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{map}_{\mathcal{C}}\left(t \times x, y\right) & \xrightarrow{(f\times \mathrm{id}_x)^*} & \mathrm{map}_{\mathcal{C}}\left(s \times x, y\right). \end{array}$$

The upper horizontal map is an equivalence for all morphisms $f \in S$ if and only if $[x,y]_{\mathcal{C}}$ is S-local, i.e. an object of $\mathcal{D}$ ([Lur09b, Proposition 5.5.4.2, (1)]). This holds for all objects $x \in \mathcal{C}$, $y \in \mathcal{D}$ if and only if (1) is satisfied. On the other hand, the lower horizontal map is an equivalence for all objects $y \in \mathcal{D}$ if and only if $f \times \mathrm{id}_x$ is an S-equivalence, i.e. a morphism in S ([Lur09b, Proposition 5.5.4.2, (2)]). This holds for all objects $x \in \mathcal{C}$ and morphisms $f \in S$ if and only if (2) is satisfied, since the cartesian product is symmetric and S is closed under composition. Thus, we obtain (1) $\Leftrightarrow$ (2).

Now, assume that (2) holds. For objects $x, x' \in \mathcal{C}$, consider the commutative triangle

$$\begin{array}{ccc} x \times x' & \xrightarrow{\eta_{x\times x'}} & L\left(x \times x'\right) \\ & \searrow^{\eta_x \times \eta_{x'}} & \downarrow \\ & & Lx \times Lx', \end{array}$$

where η is the adjunction unit, whose components are S-equivalences ([Lur09b, Proposition 5.2.7.4]), i.e. morphisms in S. That is, $\eta_{x\times x'} \in S$ and (2) implies $\eta_x \times \eta_{x'} \in S$, but S satisfies 2-out-of-3, hence $L\left(x \times x'\right) \to Lx \times Lx'$ is in S. It is a morphism between S-local objects (since $\mathcal{D}$ is stable under finite products in $\mathcal{C}$), hence an equivalence. Thus, (3) holds.

Lastly, suppose that (3) holds. If $f, g \in S$, then $L(f \times g) \simeq L(f) \times L(g)$ is a product of equivalences, hence an equivalence, and $f \times g \in S$. Thus, (2) holds. □

Corollary A.0.10. *Let $\mathcal{C}$ be a cartesian closed, presentable ∞-category, A a set of objects that generates $\mathcal{C}$ under colimits and S a small set of morphisms in $\mathcal{C}$ that is stable under $(-) \times \mathrm{id}_x$ for any $x \in A$. Then, the full subcategory $\mathcal{D} \subseteq \mathcal{C}$ of S-local objects is a strongly reflective exponential ideal.*

Proof. By [Lur09b, Proposition 5.5.4.15], $\mathcal{D} \subseteq \mathcal{C}$ is a strongly reflective subcategory and the set of morphisms inverted by the left adjoint $L\colon \mathcal{C} \to \mathcal{D}$ is precisely $\overline{S}$, the strongly saturated closure of S. To show that $\mathcal{D} \subseteq \mathcal{C}$ is an exponential ideal, it suffices by Proposition A.0.9 to show that $f \in \overline{S}$ implies $f \times \mathrm{id}_x \in \overline{S}$ for any object $x \in \mathcal{C}$.

For a fixed object $x \in \mathcal{C}$, the functor $(-) \times x\colon \mathcal{C} \to \mathcal{C}$ preserves colimits since $\mathcal{C}$ is cartesian closed,

hence the set of morphisms f such that $f \times \mathrm{id}_x \in \overline{S}$ is strongly saturated ([Lur09b, Remark 5.5.4.10]) and it suffices to verify the claim for $f \in S$. For a fixed $f \in S$, the full subcategory of $\mathcal{C}$ on those objects x such that $f \times \mathrm{id}_x \in \overline{S}$ is stable under colimits since $\mathcal{C}$ is cartesian closed and $\overline{S}$ is strongly saturated, and it contains the objects of A by assumption, hence equals $\mathcal{C}$. □

Corollary A.0.11. *Let $\mathcal{C}$ be a cartesian closed ∞-category and $\mathcal{E} \subseteq \mathcal{D} \subseteq \mathcal{C}$ reflective subcategories. If $\mathcal{D}$ is an exponential ideal in $\mathcal{C}$ (in particular, $\mathcal{D}$ is itself a cartesian closed ∞-category), then $\mathcal{E}$ is an exponential ideal in $\mathcal{D}$ if and only if $\mathcal{E}$ is an exponential ideal in $\mathcal{C}$.*

Proof. If $\mathcal{E}$ is an exponential ideal in $\mathcal{C}$, then, using Proposition A.0.3, $[x,y]_{\mathcal{D}} \simeq [x,y]_{\mathcal{C}} \in \mathcal{E}$ for any objects $x \in \mathcal{D}$, $y \in \mathcal{E}$, hence $\mathcal{E}$ is an exponential ideal in $\mathcal{D}$. The converse implication is a consequence of characterization (3) of Proposition A.0.9, since left adjoints compose. □

Proposition A.0.12. *Let $\mathcal{C}$ be a cartesian closed, presentable ∞-category and K a small simplicial set. Then,* $\mathrm{Fun}(K,\mathcal{C})$ *is a cartesian closed, presentable ∞-category.*

Proof. That $\mathrm{Fun}(K,\mathcal{C})$ is presentable is [Lur09b, Proposition 5.5.3.6]. For any $p\colon K \to \mathcal{C}$, the functor $(-) \times p\colon \mathrm{Fun}(K,\mathcal{C}) \to \mathrm{Fun}(K,\mathcal{C})$ preserves colimits since these are computed pointwise ([Lur09b, Corollary 5.1.2.3]) and $\mathcal{C}$ is cartesian closed. The adjoint functor theorem implies the result. □

Corollary A.0.13. *Under the hypotheses of Proposition A.0.12, if $\mathcal{D} \subseteq \mathcal{C}$ is an exponential ideal, then* $\mathrm{Fun}(K,\mathcal{D}) \subseteq \mathrm{Fun}(K,\mathcal{C})$ *is an exponential ideal.*

Proof. This is immediate by characterization (3) of Proposition A.0.9, using [Lan21, Proposition 5.1.16] and that limits in functor categories are computed pointwise. □

Lemma A.0.14. *Let $\mathcal{C}$ be a cartesian closed ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a reflective subcategory that generates $\mathcal{C}$ under colimits and is a cartesian closed ∞-category. Then, $\mathcal{D} \subseteq \mathcal{C}$ is an exponential ideal.*

Proof. For objects $x, y \in \mathcal{D}$, the counit morphism $[x,y]_{\mathcal{D}} \times x \to y$ induces a morphism $[x,y]_{\mathcal{D}} \to [x,y]_{\mathcal{C}}$ in $\mathcal{C}$. Then, for any object $t \in \mathcal{D}$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{D}}\left(t, [x,y]_{\mathcal{D}}\right) & \longrightarrow & \mathrm{map}_{\mathcal{C}}\left(t, [x,y]_{\mathcal{C}}\right) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{map}_{\mathcal{D}}\left(t \times x, y\right) & = & \mathrm{map}_{\mathcal{C}}\left(t \times x, y\right). \end{array}$$

The full subcategory of $\mathcal{C}$ on those objects $t \in \mathcal{C}$ such that $\mathrm{map}_{\mathcal{D}}\left(t, [x,y]_{\mathcal{D}}\right) \to \mathrm{map}_{\mathcal{C}}\left(t, [x,y]_{\mathcal{C}}\right)$ is an equivalence thus contains $\mathcal{D}$ and is stable under colimits by the dual of [Lur09b, Proposition 5.1.3.2]. The assumption implies that it is equal to $\mathcal{C}$, hence $[x,y]_{\mathcal{D}} \xrightarrow{\sim} [x,y]_{\mathcal{C}}$ is an equivalence.

Now, fix an object $y \in \mathcal{D}$ and consider the full subcategory of $\mathcal{C}$ on those objects $x \in \mathcal{C}$ such that $[x, y]_{\mathcal{C}} \in \mathcal{D}$. It contains $\mathcal{D}$ by the preceding argument and is stable under colimits since $[-, -]$ preserves limits in the first variable by Proposition A.0.6 and $\mathcal{D}$ is stable under limits. The assumption implies that it is equal to $\mathcal{C}$, hence $\mathcal{D} \subseteq \mathcal{C}$ is an exponential ideal. □

Remark A.0.15. This is an ∞-categorical generalization of [Str80, Theorem 3.10].

Proposition A.0.16. *For an ∞-category $\mathcal{C}$, the following conditions are equivalent:*

(1) The ∞-category $\mathcal{C}$ is cartesian closed and presentable.

(2) There exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is equivalent to a strongly reflective exponential ideal in $\mathcal{P}(\mathcal{D})$.

Proof. The implication (2) $\Rightarrow$ (1) is a consequence of Proposition A.0.3 and [Lur09b, Remark 5.5.1.6]. For the implication (1) $\Rightarrow$ (2), suppose that $\mathcal{C}$ is a cartesian closed, presentable ∞-category. If κ is a regular cardinal such that $\mathcal{C}$ is κ-accessible, then the proof of [Lur09b, Theorem 5.5.1.1] shows that there is a small (idempotent complete) ∞-category $\mathcal{D}$ such that $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{D})$ and $\mathrm{Ind}_\kappa(\mathcal{D}) \subseteq \mathcal{P}(\mathcal{D})$ is strongly reflective. The assumption implies that $\mathrm{Ind}_\kappa(\mathcal{D}) \subseteq \mathcal{P}(\mathcal{D})$ is cartesian closed and it generates $\mathcal{P}(\mathcal{D})$ under colimits as it contains the essential image of the Yoneda embedding ([Lur09b, Corollary 5.1.5.8]), so it is an exponential ideal by Lemma A.0.14. □

Proposition A.0.17. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a cartesian closed functor of cartesian closed ∞-categories and $\mathcal{D}^0 \subseteq \mathcal{D}$ an exponential ideal. Then, $\mathcal{C}^0 = f^{-1}(\mathcal{D}^0) \subseteq \mathcal{C}$ is an exponential ideal.*

Proof. For objects $x \in \mathcal{C}$ and $y \in \mathcal{C}^0$, we have $f\,[x, y]_{\mathcal{C}} \simeq [fx, fy]_{\mathcal{D}} \in \mathcal{D}^0$ since f is cartesian closed and $fy \in \mathcal{D}^0$, hence $[x, y]_{\mathcal{C}} \in \mathcal{C}^0$. □

B Sequential Limits and Adjunctions

In this Appendix, we present a general construction of adjunctions between sequential limits of ∞-categories. To this end, suppose that

$$\mathcal{C} \longrightarrow \ldots \xrightarrow{p_{n+1}} \mathcal{C}_{n+1} \xrightarrow{p_n} \mathcal{C}_n \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_0} \mathcal{C}_0$$

is a sequential limit diagram in $\mathbf{CAT}_\infty$. This diagram is indexed by the infinite spine $(\mathrm{Sp}^\infty)^{op}$, though it can be equivalently understood to be indexed by $N(\mathbb{N})^{op}$. The limit $\mathcal{C}$ can be identified by inductively applying the factorization into $\mathcal{C} \xhookrightarrow{\sim} \mathrm{Iso}(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C} \twoheadrightarrow \mathcal{D}$ of a map $f\colon \mathcal{C} \to \mathcal{D}$ as a trivial cofibration followed by a fibration to obtain an equivalent fibrant diagram, then taking its strict limit. In other words, $\mathcal{C}$ is (equivalent to) the ∞-category whose objects are given by a collection of objects $c_n \in \mathcal{C}_n$

for all $n \geq 0$ and equivalences $c_n \xrightarrow{\sim} p_n c_{n+1}$ in $\mathcal{C}_n$ for all $n \geq 0$. Let us also introduce the notational shorthands $p_n^k = p_n \dots p_{n+k-1}$ for $k \geq 0$ and $p_n^\infty \colon \mathcal{C} \to \mathcal{C}_n$ for the canonical projection.

Proposition B.0.1. *In the above scenario, there is a canonical equivalence*

$$\mathrm{map}_{\mathcal{C}}\left(c, c'\right) \xrightarrow{\sim} \lim_{n \geq 0} \mathrm{map}_{\mathcal{C}_n}\left(p_n^\infty c, p_n^\infty c'\right), \qquad \forall c, c' \in \mathcal{C}.$$

Proof. This is clear in the above strict model, using [Lan21, Lemma 3.1.11]. □

Corollary B.0.2. *The sequential limit of fully faithful functors is fully faithful.* □

Remark B.0.3. This can also be generalized to diagrams of arbitrary shape.

The following arguments are ∞-categorical generalizations of the arguments in Subsection 2.3, in particular the proof of Theorem 2.3.6.

Theorem B.0.4. *Let a natural transformation of sequential limit diagrams of ∞-categories*

$$\begin{array}{ccccccccccc}
\mathcal{C} & \longrightarrow & \dots & \xrightarrow{p_{n+1}} & \mathcal{C}_{n+1} & \xrightarrow{p_n} & \mathcal{C}_n & \xrightarrow{p_{n-1}} & \dots & \xrightarrow{p_0} & \mathcal{C}_0 \\
\downarrow{\scriptstyle R} & & & & \downarrow{\scriptstyle R_{n+1}} & \swarrow_{\sim} & \downarrow{\scriptstyle R_n} & & & & \downarrow{\scriptstyle R_0} \\
\mathcal{D} & \longrightarrow & \dots & \xrightarrow{q_{n+1}} & \mathcal{D}_{n+1} & \xrightarrow{q_n} & \mathcal{D}_n & \xrightarrow{q_{n-1}} & \dots & \xrightarrow{q_0} & \mathcal{D}_0
\end{array}$$

be given, in which the functors $R_n \colon \mathcal{C}_n \to \mathcal{D}_n$ admit left adjoints $L_n \colon \mathcal{D}_n \to \mathcal{C}_n$ and the transfer maps p_n, q_n, $n \geq 0$ preserve sequential colimits. Then, $R \colon \mathcal{C} \to \mathcal{D}$ admits a left adjoint, given by

$$L \colon \mathcal{D} \to \mathcal{C},\ d \mapsto \left(\operatorname*{colim}_{k \geq 0} p_n^k L_{n+k} q_{n+k}^\infty d \right)_{n \geq 0}.$$

If the above squares are vertically left adjointable, this simplifies to $L \colon \mathcal{D} \to \mathcal{C}$, $d \mapsto (L_n q_n^\infty d)_{n \geq 0}$.

Remark B.0.5. The case in which the above squares are vertically left adjointable generalizes to diagrams of arbitrary shape. The full Theorem, however, has much weaker assumptions and instead exploits the specific shape of the diagram, e.g. it does not hold for pullback diagrams.

Proof. Taking fibrant replacements and using homotopy lifting, we may assume w.l.o.g. that the transfer maps are isofibrations, the squares commute strictly and the sequential limits are modeled as strict limits, as in the introductory discussion.

For $n \geq 0$, there is a natural transformation $\alpha_n \colon L_n q_n \Rightarrow p_n L_{n+1}$, induced by the composite $q_n \Rightarrow q_n R_{n+1} L_{n+1} = R_n p_n L_{n+1}$ of q_n and the unit transformation η_{n+1} of $L_{n+1} \dashv R_{n+1}$ under the adjunction. Let $\mathrm{Sp}_{\geq n}^\infty = \mathrm{Sp}^\infty \cap N\left(\mathbb{N}_{\geq n}\right)$ and consider the diagram $\mathrm{Sp}_{\geq n}^\infty \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}_n)$ depicted as

$$L_n q_n^\infty \xrightarrow{\alpha_n q_{n+1}^\infty} p_n L_{n+1} q_{n+1}^\infty \xrightarrow{p_n \alpha_{n+1} q_{n+2}^\infty} p_n^2 L_{n+2} q_{n+2}^\infty \longrightarrow \dots .$$

There is a strictly commutative diagram

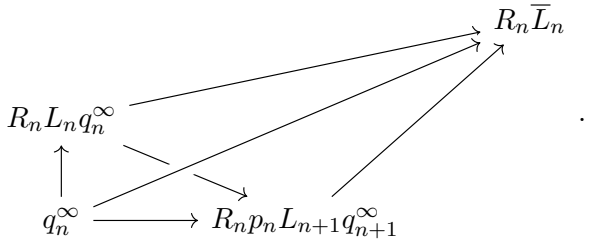
$$\begin{array}{ccc} \mathrm{Sp}^{\infty}_{\geq n+1} & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathcal{C}_{n+1}) \\ \downarrow & & \downarrow{\scriptstyle (p_n)_*} \\ \mathrm{Sp}^{\infty}_{\geq n} & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathcal{C}_n), \end{array}$$

the maps $(p_n)_*$ are inner fibrations and the inclusions $\mathrm{Sp}^{\infty}_{\geq n} \hookrightarrow N(\mathbb{N}_{\geq n})$ are inner anodyne ([Lur18, Proposition 1.5.7.3]), so we can inductively pick extensions $N(\mathbb{N}_{\geq n}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}_n)$ compatible with the $(p_n)_*$, $n \geq 0$. These define, by currying, the left triangle in the strictly commutative diagram

$$\begin{array}{ccccccc} \mathcal{D} & \longrightarrow & \mathrm{Fun}\left(N\left(\mathbb{N}_{\geq n+1}\right), \mathcal{C}_{n+1}\right) & \xleftarrow{\sim} & \mathrm{Fun}\left(N\left(\overline{\mathbb{N}}_{\geq n+1}\right), \mathcal{C}_{n+1}\right)_{\mathrm{colim}} & \longrightarrow & \mathcal{C}_{n+1} \\ & \searrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle p_n} \\ & & \mathrm{Fun}\left(N\left(\mathbb{N}_{\geq n}\right), \mathcal{C}_n\right) & \xleftarrow{\sim} & \mathrm{Fun}\left(N\left(\overline{\mathbb{N}}_{\geq n}\right), \mathcal{C}_n\right)_{\mathrm{colim}} & \longrightarrow & \mathcal{C}_n, \end{array}$$

where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the cone on $\mathbb{N}$, the middle terms are the full subcategories on colimit diagrams and the leftwards horizontal restriction maps are trivial fibrations ([Lur18, Proposition 7.1.6.3]). The middle vertical map exists since $N(\mathbb{N}_{\geq n+1}) \hookrightarrow N(\mathbb{N}_{\geq n})$ is cofinal and p_n preserves sequential colimits. They are isofibrations, so we may inductively choose compatible sections of the horizontal equivalences and obtain horizontal composites $\overline{L}_n \colon \mathcal{D} \to \mathcal{C}_n$, given on objects by $\overline{L}_n(d) = \mathrm{colim}_{m \geq n} p_n^{m-n} L_m q_m^{\infty} d$ and compatible with the projections p_n, $n \geq 0$. These induce the candidate left adjoint $L \colon \mathcal{D} \to \mathcal{C}$.

The composite $\mathcal{D} \to \mathrm{Fun}\left(N\left(\overline{\mathbb{N}}_{\geq n}\right), \mathcal{C}_n\right) \to \mathrm{Fun}\left(\Delta^1, \mathcal{C}_n\right)$ obtained by restricting along the edge $n \to \infty$ curries to a natural transformation $\mathcal{D} \times \Delta^1 \to \mathcal{C}_n$. This induces, post-composing with R_n, a natural transformation $R_n L_n q_n^{\infty} \Rightarrow R_n \overline{L}_n = q_n^{\infty} RL$. On the other hand, the unit η_n of $L_n \dashv R_n$ induces, pre-composing with q_n^{∞}, a natural transformation $q_n^{\infty} \Rightarrow R_n L_n q_n^{\infty}$. This yields inner horns $\Lambda^2_1 \to \mathrm{Fun}\left(\mathcal{D}, \mathcal{D}_n\right)$, which we will inductively extend to 2-simplices compatible with the projections q_n, $n \geq 0$. The base case is arbitrary, and having constructed a 2-simplex $\Delta^2 \to \mathrm{Fun}(\mathcal{D}, \mathcal{D}_n)$, it defines the 2-nd face of a 3-simplex



.

The 0-th face is given by the composite $\Delta^2 \longrightarrow N(\mathbb{N}_{\geq n})^{op} \longrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{C}_n) \xrightarrow{(R_n)_*} \mathrm{Fun}(\mathcal{D}, \mathcal{D}_n)$, where

the 2-simplex picks out $n \to n+1 \to \infty$. The 3-rd face can be filled by the construction of α_n. Thus, we have constructed an inner horn $\Lambda^3_1 \to \mathrm{Fun}(\mathcal{D}, \mathcal{D}_n)$, which can be extended to a 3-simplex $\Delta^3 \to \mathrm{Fun}(\mathcal{D}, \mathcal{D}_n)$. Its 1-st face defines the bottom map in the lifting problem

$$\begin{array}{ccc} \Lambda^2_1 & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathcal{D}_{n+1}) \\ \downarrow & \nearrow & \downarrow{\scriptstyle (q_n)_*} \\ \Delta^2 & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathcal{D}_n), \end{array}$$

which can be solved as $(q_n)_*$ is an inner fibration. This completes the induction and we obtain natural transformations $q_n^\infty \Rightarrow q_n^\infty RL$ compatible with the projections q_n, $n \geq 0$, hence a natural transformation $\eta\colon \mathrm{id}_{\mathcal{D}} \Rightarrow RL$, which we now show to be a unit transformation.

For any objects $c = (c_n)_{n\geq 0} \in \mathcal{C}$ and $d = (d_n)_{n\geq 0} \in \mathcal{D}$, these constructions ensure that there is a commutative diagram in the homotopy category of spaces

$$\begin{array}{ccc} \lim\limits_{n\geq 0} \mathrm{map}_{\mathcal{C}_n}\left(\operatorname*{colim}\limits_{m\geq n} p_n^{m-n} L_m d_m, c_n\right) & \longrightarrow & \lim\limits_{n\geq 0} \mathrm{map}_{\mathcal{D}_n}\left(R_n \operatorname*{colim}\limits_{m\geq n} p_n^{m-n} L_m d_m, R_n c_n\right) \\ \downarrow & & \downarrow \\ \lim\limits_{n\geq 0} \mathrm{map}_{\mathcal{C}_n}(L_n d_n, c_n) & \longrightarrow & \lim\limits_{n\geq 0} \mathrm{map}_{\mathcal{D}_n}(R_n L_n d_n, R_n c_n) \\ & \searrow^{\sim} & \downarrow \\ & & \lim\limits_{n\geq 0} \mathrm{map}_{\mathcal{D}_n}(d_n, R_n c_n). \end{array} \tag{6}$$

The right-down composite is identified by Proposition B.0.1 with the map

$$\mathrm{map}_{\mathcal{C}}(Ld, c) \xrightarrow{R} \mathrm{map}_{\mathcal{D}}(RLd, Rc) \xrightarrow{(\eta_d)^*} \mathrm{map}_{\mathcal{D}}(d, Rc).$$

The diagonal map in (6) is an equivalence since $L_n \dashv R_n$ is an adjunction for all $n \geq 0$, so it remains to demonstrate that the left vertical map in (6) is an equivalence. To this end, let P denote the poset $\{(n, m) \in \mathbb{N}^2 | m \geq n\}$, regarded as a category, and consider a strict diagram $N(P)^{op} \to \mathbf{Spc}$, given on objects by $(n, m) \mapsto \mathrm{map}_{\mathcal{C}_n}(p_n^{m-n} L_m d_m, c_n)$. We calculate

$$\begin{aligned} \lim_{N(\mathbb{N})^{op}} \mathrm{map}_{\mathcal{C}_n}\left(\operatorname*{colim}_{N(\mathbb{N}_{\geq n})} p_n^{m-n} L_m d_m, c_n\right) &\simeq \lim_{N(\mathbb{N})^{op}} \lim_{N(\mathbb{N}_{\geq n})^{op}} \mathrm{map}_{\mathcal{C}_n}\left(p_n^{m-n} L_m d_m, c_n\right) \\ &\simeq \lim_{N(P)^{op}} \mathrm{map}_{\mathcal{C}_n}\left(p_n^{m-n} L_m d_m, c_n\right) \\ &\simeq \lim_{N(\mathbb{N})^{op}} \mathrm{map}_{\mathcal{C}_n}\left(L_n d_n, c_n\right). \end{aligned}$$

The first equivalence holds since $\mathrm{map}_{\mathcal{C}_n}(-, c_n)$ preserves small limits by [Lur09b, Proposition 5.1.3.2]. The second equivalence holds by combining [Lur18, Proposition 7.3.4.1] and [Lur09b, Lemma 4.3.2.7] since $N(P) \to N(\mathbb{N})$, $(n, m) \mapsto n$ is a cocartesian fibration with fiber $N(\mathbb{N}_{\geq n})^{op}$ over n, $n \geq 0$. The last equivalence holds since $N(\mathbb{N})^{op} \hookrightarrow N(P)^{op}$, $n \mapsto (n, n)$ is coinitial by [Lur09b, Theorem 4.1.3.1], since $\mathbb{N} \times_P P_{(n,m)/} \cong \mathbb{N}_{m/} \cong \mathbb{N}_{\geq m}$ has weakly contractible nerve for all $(n, m) \in P$. □

Corollary B.0.6. *Let*

$$\mathcal{C} \longrightarrow \ldots \xrightarrow{R_{n+1}} \mathcal{C}_{n+1} \xrightarrow{R_n} \mathcal{C}_n \xrightarrow{R_{n-1}} \ldots \xrightarrow{R_0} \mathcal{C}_0$$

be a sequential limit diagram of ∞-categories, in which the functors $R_n \colon \mathcal{C}_{n+1} \to \mathcal{C}_n$ admit left adjoints $L_n \colon \mathcal{C}_n \to \mathcal{C}_{n+1}$ and preserve sequential colimits. Then, $R_0^\infty \colon \mathcal{C} \to \mathcal{C}_0$ admits a left adjoint

$$L \colon \mathcal{C}_0 \to \mathcal{C},\ c \mapsto \left(\operatorname*{colim}_{k \geq 0} R_n^k L_0^{n+k} c \right)_{n \geq 0}.$$

If the left adjoints L_n, $n \geq 0$ are fully faithful, then L is fully faithful and given by $c \mapsto (L_0^n c)_{n \geq 0}$.

Proof. This follows by applying Theorem B.0.4 to the diagram

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{R_{n+1}} & \mathcal{C}_{n+1} & \xrightarrow{R_n} & \mathcal{C}_n & \xrightarrow{R_{n-1}} & \ldots & \xrightarrow{R_0} & \mathcal{C}_0 \\
 & & \downarrow{\scriptstyle R_0^{n+1}} & & \downarrow{\scriptstyle R_0^n} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
\ldots & \xrightarrow{\mathrm{id}} & \mathcal{C}_0 & \xrightarrow{\mathrm{id}} & \mathcal{C}_0 & \xrightarrow{\mathrm{id}} & \ldots & \xrightarrow{\mathrm{id}} & \mathcal{C}_0.
\end{array}$$

If the left adjoints L_n, $n \geq 0$ are fully faithful, the squares are vertically left adjointable, so the formula simplifies and L is fully faithful by Corollary B.0.2. □

Remark B.0.7. The case in which the left adjoints are fully faithful can also be proven directly.

Corollary B.0.8. *Let*

$$\mathcal{C} \longrightarrow \ldots \xrightarrow{R_{n+1}} \mathcal{C}_{n+1} \xrightarrow{R_n} \mathcal{C}_n \xrightarrow{R_{n-1}} \ldots \xrightarrow{R_0} \mathcal{C}_0$$

be a sequential limit diagram of ∞-categories, in which the functors $R_n \colon \mathcal{C}_{n+1} \to \mathcal{C}_n$ admit left adjoints $L_n \colon \mathcal{C}_n \to \mathcal{C}_{n+1}$ and preserve sequential colimits. Then, there is a canonical equivalence

$$\operatorname*{colim}_{k \geq 0} L_k^\infty R_k^\infty \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}.$$

Proof. To see this, consider the diagram

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\mathrm{id}} & \mathcal{C} & \xrightarrow{\mathrm{id}} & \mathcal{C} & \xrightarrow{\mathrm{id}} & \dots & \xrightarrow{\mathrm{id}} & \mathcal{C} \\ & & \downarrow{\scriptstyle R^{\infty}_{n+1}} & & \downarrow{\scriptstyle R^{\infty}_{n}} & & & & \downarrow{\scriptstyle R^{\infty}_{0}} \\ \dots & \xrightarrow{R_{n+1}} & \mathcal{C}_{n+1} & \xrightarrow{R_n} & \mathcal{C}_n & \xrightarrow{R_{n-1}} & \dots & \xrightarrow{R_0} & \mathcal{C}_0. \end{array}$$

The vertical projections admit left adjoints by applications of Corollary B.0.6. Then, applying Theorem B.0.4 to this diagram constructs a left adjoint to $\mathrm{id}_{\mathcal{C}}$, i.e. $\mathrm{id}_{\mathcal{C}}$ itself. □

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