Notes on Sites and Sheaves

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These notes are intended to serve as a quick (re-)collection on the ∞-categorical theory of sheaves of anima on a site. They originated as background material for a talk I gave in the *Kleine Topologie Seminar* at the University of Münster in the Winter of 2025 [Gei25], but grew into their own thing.

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This material is primarily based on parts of [Lur09, Section 6.2] and [Lur18, Appendix A], as well as some folklore. Furthermore, some of the ideas below have been independently developed in [Bas26].

Contents

1 Sheaves on a Site **1**

2 A Class of Convenient Sites **4**

3 Sheaves and Slices **9**

4 The Convenient Sites are no Loss of Generality **11**

1 Sheaves on a Site

Let $\mathcal{C}$ be a small ∞-category. A *sieve* on an object c of $\mathcal{C}$ is a collection of maps in $\mathcal{C}$ with target c that is stable under precomposition with arbitrary maps (in particular, stable under homotopy). This can be interpreted either as a full subcategory $U_{/c} \subseteq \mathcal{C}_{/c}$ satisfying this condition or as a monomorphism $U \hookrightarrow j(c)$ of presheaves[(1)] on $\mathcal{C}$ (for every object c' of c, the sub-anima $U(c') \hookrightarrow \mathrm{Map}_{\mathcal{C}}(c', c)$ consists of those components belonging to the sieve). The *trivial sieve* on c is given by the identity $\mathrm{id}_{j(c)}$. Furthermore, if $f\colon d \to c$ is any morphism in $\mathcal{C}$, then we may form the pullback

$$\begin{array}{ccc} f^*U & \hookrightarrow & j(d) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle j(f)} \\ U & \hookrightarrow & j(c)\,. \end{array}$$

[(1)] The Yoneda embedding will always be denoted by $j\colon \mathcal{C} \hookrightarrow \mathbf{PSh}(\mathcal{C})$.

The sieve $f^*U \hookrightarrow j(d)$ (monomorphisms are stable under pullback in any ∞-category) consists precisely of all those morphisms $\varphi\colon d' \to d$ such that $f\varphi$ belongs to U. These two perspectives on sieves are further related by the following lemma.

Lemma 1.1. *For $\mathcal{C}$ a small ∞-category, $U \hookrightarrow j(c)$ a sieve and $U_{/c} \subseteq \mathcal{C}_{/c}$ the corresponding full subcategory, there is a natural equivalence $\lim_{(d\to c)\in U_{/c}} \mathcal{F}(d) \simeq \mathrm{Map}_{\mathbf{PSh}(\mathcal{C})}(U, \mathcal{F})$ for any $\mathcal{F} \in \mathbf{PSh}(\mathcal{C})$.*

Proof. This follows by mapping into $\mathcal{F}$ the canonical equivalence $\mathrm{colim}_{(d\to c)\in U_{/c}} j(d) \xrightarrow{\sim} U$, which is an instance of density since $U_{/c}$ is precisely the slice category of representables mapping to U. □

A *Grothendieck topology* τ on a small ∞-category $\mathcal{C}$ is a collection of sieves called *covering sieves* on any object c that contain the trivial sieve, are stable under pullback in the above sense and satisfy the following 'locality' condition: for any sieve $V \hookrightarrow j(c)$ and covering sieve $U \hookrightarrow j(c)$, if for any $f\colon c' \to c$ in U, the pullback $f^*V \hookrightarrow j(c')$ is a covering sieve, then $V \hookrightarrow j(c)$ is a covering sieve.

Let $(\mathcal{C}, \tau)$ be an ∞-*site*, i.e. a pair consisting of a small ∞-category equipped with a Grothendieck topology.[(2)] A presheaf $\mathcal{F}\colon \mathcal{C}^{op} \to \mathrm{An}$ is called a *sheaf* (with respect to τ) if $\mathcal{F}(c) \to \lim_{(d\to c)\in U_{/c}} \mathcal{F}(d)$ or equivalently $\mathcal{F}(c) \simeq \mathrm{Map}_{\mathbf{PSh}(C)}(j(c), \mathcal{F}) \to \mathrm{Map}_{\mathbf{PSh}(C)}(U, \mathcal{F})$ is an equivalence for any covering sieve $U \hookrightarrow j(c)$. The full subcategory on sheaves is denoted $\mathbf{Sh}(\mathcal{C}, \tau) \subseteq \mathbf{PSh}(\mathcal{C})$. The first description implies that this subcategory is stable under small limits and κ-filtered colimits for a sufficiently large regular cardinal κ since the sheaf condition only involves limits of a bounded size. The second description implies that this inclusion is a Bousfield localization [Lur09, Proposition 5.5.4.15], in particular admits a left adjoint $(-)^{\#} : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, \tau)$ called *sheafification*. It is possible to prove that sheafification is left-exact—either by using the plus construction as in [Lur09, Proposition 6.2.2.7] or using modalities as in [Hoy26, Theorem 5.2]—and deduce the following.

Theorem 1.2. *The ∞-category $\mathbf{Sh}(\mathcal{C}, \tau)$ is an ∞-topos.* □

To describe Grothendieck topologies more efficiently, we define the notion of a *Grothendieck pretopology* on a small ∞-category $\mathcal{C}$ that admits pullbacks. This is a collection of families $(U_i \to U)_{i\in I}$ called *covers* on any object U that contain equivalences $(f\colon V \xrightarrow{\sim} U)$, s.t. if $(U_i \to U)_{i\in I}$ is a cover of U and $f\colon V \to U$ is a morphism, then $(U_i \times_U V \to V)_{i\in I}$ is a cover of V and, if $(U_i \to U)_{i\in I}$ is a cover of U and furthermore $(V_{ij} \to U_i)_{j\in J_i}$ is a cover of U_i for all $i \in I$, then $(V_{ij} \to U_i \to U)_{(i,j)\in\coprod_{i\in I} J_i}$ is a cover of U. The concepts of covers and covering sieves are related by the following Lemma.

Lemma 1.3. *Let $\mathcal{C}$ be a small ∞-category and $\mathfrak{U} = (U_i \to U)_{i\in I}$ a small collection of morphisms in $\mathcal{C}$. This induces a (surjection[(3)],mono)-factorization $\coprod_{i\in I} j(U_i) \twoheadrightarrow \left|\check{C}(\mathfrak{U})\right| \hookrightarrow j(U)$ in $\mathbf{PSh}(\mathcal{C})$ via the Čech nerve*

$$\check{C}(\mathfrak{U}) = \left([n] \mapsto \coprod_{i_0,\dots,i_n\in I} U_{i_0} \times_U \dots \times_U U_{i_n} \right).$$

The monomorphism $\left|\check{C}(\mathfrak{U})\right| \hookrightarrow j(U)$ describes the sieve generated by $\mathfrak{U}$ *and consists precisely of those morphisms with target U that factor through one of the $U_i \to U$, $i \in I$.* □

[(2)] The Grothendieck topology will often be dropped in notation.

[(3)] i.e. 0-connective in the corresponding ∞-topos. These are called 'effective epimorphisms' in [Lur09], but that is terrible terminology because they do not need to be epimorphisms.

Let $\mathcal{C}$ be a small ∞-category admitting pullbacks and ρ a Grothendieck pretopology on $\mathcal{C}$. A presheaf $\mathcal{F}\colon \mathcal{C}^{op} \to \mathrm{An}$ is called a *sheaf* (with respect to ρ) if, for any ρ-cover $\mathfrak{U} = (U_i \to U)_{i\in I}$, it is local w.r.t. the sieve $\left|\check{C}\left(\mathfrak{U}\right)\right| \hookrightarrow j\left(U\right)$, i.e.

$$\mathcal{F}\left(U\right) \longrightarrow \prod_{i_0\in I} \mathcal{F}\left(U_{i_0}\right) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \prod_{i_0,i_1\in I} \mathcal{F}\left(U_{i_0}\times_U U_{i_1}\right) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \dots$$

is a limit diagram in An. This is the homotopy-coherent analogue of the usual equalizer diagram describing the sheaf condition for a cover. Now, any Grothendieck pretopology ρ on $\mathcal{C}$ induces a Grothendieck topology $\overline{\rho}$ on $\mathcal{C}$ whose covering sieves are all those sieves that contain a sieve generated by a ρ-cover. In the converse direction, a Grothendieck topology induces a Grothendieck pretopology whose covers are all those families of morphisms that generate a covering sieve. The resulting theory of sheaves is the same by the next proposition.

Proposition 1.4. *Let $\mathcal{C}$ be a small ∞-category that admits pullbacks and ρ a Grothendieck pretopology on $\mathcal{C}$. Then, a presheaf on $\mathcal{C}$ is a sheaf with respect to ρ if and only if it is a sheaf with respect to $\overline{\rho}$.*

Proof. It is clear that the sheaf condition with respect to $\overline{\rho}$ implies the sheaf condition with respect to ρ. In the converse direction, assume that $\mathcal{F}$ is a sheaf with respect to ρ and let $U \hookrightarrow j\left(c\right)$ be a sieve containing the sieve generated by a ρ-cover $\mathfrak{U}$. To show that $\mathcal{F}$ is local w.r.t. $U \hookrightarrow j\left(c\right)$, it suffices to show that the diagram $\left(U_{/c}\right)^{op} \to \mathrm{An}$, $(c' \to c) \mapsto \mathcal{F}\left(c'\right)$ is right Kan extended from its restriction to $\left(\left|\check{C}\left(\mathfrak{U}\right)\right|_{/c}\right)^{op}$. This follows from the assumption on $\mathcal{F}$ and the pointwise formula for Kan extension by the canonical identification of relative slices $\left(U_{/c}\right)_{/f} \times_{U_{/c}} \left|\check{C}\left(\mathfrak{U}\right)\right|_{/c} \cong \left|\check{C}\left(f^*\mathfrak{U}\right)\right|_{/d}$ for any $f\colon d \to c$ in U. □

Example 1.5. Let X be a topological space and $\mathrm{Open}(X)$ be the poset of its open subsets. This is a category that admits pullbacks, equipped with the Grothendieck topology given by open covers in the usual sense. Then, a presheaf $\mathcal{F}\colon \mathrm{Open}(X)^{op} \to \mathrm{An}$ is a sheaf with respect to this Grothendieck topology if and only if it satisfies

a) $\mathcal{F}(\emptyset) \simeq *$,

b) $\mathcal{F}(U \cup V) \simeq \mathcal{F}(U) \times_{\mathcal{F}(U\cap V)} \mathcal{F}(V)$ for all open $U, V \subseteq X$ and

c) $\mathcal{F}(\bigcup_{i\in I}) \simeq \lim_{i\in I} \mathcal{F}(U_i)$ for any directed poset I and diagram $I \to \mathrm{Open}(X)$, $i \mapsto U_i$.

In one direction, this is clear: a) is the sheaf condition for the empty cover of the empty open, b) is the sheaf condition for the open cover $\mathfrak{U} = (U, V)$ since $\left|\check{C}(\mathfrak{U})\right| \simeq j(U) \sqcup_{j(U\cap V)} j(V)$, and c) is the sheaf condition for the open cover $\mathfrak{U} = (U_i)_{i\in I}$ since $\left|\check{C}\left(\mathfrak{U}\right)\right| \simeq \mathrm{colim}_{i\in I}\, j(U_i)$. In the converse direction, let $\mathfrak{U} = (U_i)_{i\in I}$ be an arbitrary open cover of $U \subseteq X$. Let $\mathcal{P}_f(I)$ denote the set of finite subsets of I and note that $\coprod_{i\in I} j(U_i) \simeq \mathrm{colim}_{J\in\mathcal{P}_f(I)} \coprod_{i\in J} j(U_i)$, which implies that $\left|\check{C}\left(\mathfrak{U}\right)\right| \simeq \mathrm{colim}_{J\in\mathcal{P}_f(I)} \left|\check{C}\left(\mathfrak{U}|_J\right)\right|$ using that $\mathcal{P}_f(I)$ is filtered and filtered colimits commute with pullbacks in An. Thus, the monomorphism $\left|\check{C}\left(\mathfrak{U}\right)\right| \hookrightarrow j(U)$ factors as

$$\left|\check{C}\left(\mathfrak{U}\right)\right| \simeq \operatorname*{colim}_{J\in\mathcal{P}_f(I)} \left|\check{C}\left(\mathfrak{U}|_J\right)\right| \hookrightarrow \operatorname*{colim}_{J\in\mathcal{P}_f(I)} j\left(\bigcup_{i\in J} U_i\right) \hookrightarrow j(U).$$

The assumption c) guarantees that $\mathcal{F}$ is local w.r.t. the latter morphism and thus, pulling out the colimit, we may assume w.l.o.g. that $\mathfrak{U} = (U_1, \dots, U_n)$ is a finite cover. In the case $n = 0$, the sheaf condition for this cover

is a). In the case $n = 1$, the sheaf condition is tautological. In the case $n = 2$, the sheaf condition for this cover is b) and, lastly, in the case $n > 2$, the sheaf condition reduces to the previous one by an induction.

For later use, we record a description of surjections in sheaf topoi.

Proposition 1.6. *Let $(\mathcal{C}, \tau)$ be an ∞-site. A map $\alpha\colon \mathcal{F} \to \mathcal{G}$ in $\mathbf{Sh}(\mathcal{C}, \tau)$ is surjective if and only if, for any $U \in \mathcal{C}$ and morphism $j(U) \to \mathcal{G}$, there exists a cover $(U_i \to U)_{i \in I}$ and a commutative diagram*

$$\begin{array}{ccc} \bigsqcup_{i \in I} j(U_i) & \dashrightarrow & \mathcal{F} \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ j(U) & \longrightarrow & \mathcal{G}. \end{array}$$

Proof. The 0-truncated objects in $\mathbf{Sh}(\mathcal{C}, \tau)$ identify as

$$\tau_{\leq 0}\mathbf{Sh}(\mathcal{C}, \tau) = \mathbf{Sh}(\mathcal{C}, \tau) \cap \tau_{\leq 0}\mathbf{PSh}(\mathcal{C}) = \mathbf{Sh}(\mathcal{C}, \tau) \cap \mathbf{PSh}(\mathcal{C}; \mathrm{Set}) = \mathbf{Sh}((\mathcal{C}, \tau); \mathrm{Set}) = \mathbf{Sh}((h\mathcal{C}, \tau); \mathrm{Set}).$$

The Postnikov-truncation $\tau_{\leq 0}\colon \mathbf{Sh}(\mathcal{C}, \tau) \to \mathbf{Sh}((h\mathcal{C}, \tau); \mathrm{Set})$ is given by the pointwise 0-truncation of anima, followed by sheafification. Thus, by [Lur09, Proposition 7.2.1.14], the map $\alpha\colon \mathcal{F} \to \mathcal{G}$ is surjective if and only if $(\tau_{\leq 0}\alpha)^{\#} : (\tau_{\leq 0}\mathcal{F})^{\#} \to (\tau_{\leq 0}\mathcal{G})^{\#}$ is an epimorphism and this is the case if and only if $\tau_{\leq 0}\alpha\colon \tau_{\leq 0}\mathcal{F} \to \tau_{\leq 0}\mathcal{G}$ is locally surjective, which unwinds to the given characterization. □

2 A Class of Convenient Sites

In this section, we introduce a particularly convenient class of sites.

Proposition 2.1 ([Lur18, Propositions A.3.2.1, A.3.3.1 & A.5.7.2])**.** *Let $\mathcal{C}$ be a small ∞-category that admits pullbacks as well as finite coproducts that are universal and disjoint. Let S be a class of morphisms in $\mathcal{C}$ that contains the equivalences, is closed under composition and stable under pullbacks as well as finite coproducts. Then:*

1. *The collection of covers $(U_i \to U)_{i=1,\dots,n}$ such that $(\bigsqcup_{i=1}^n U_i \to U) \in S$ defines a Grothendieck pretopology on $\mathcal{C}$.*
2. *A presheaf $\mathcal{F}\colon \mathcal{C}^{op} \to \mathrm{An}$ is a sheaf with respect to this Grothendieck pretopology if and only if*
   - *$\mathcal{F}$ preserves finite products and*
   - *(Čech descent) $\mathcal{F}$ takes Čech nerves of morphisms in S to limit diagrams in* An*, i.e.*

$$\mathcal{F}(Y) \longrightarrow \mathcal{F}(X) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathcal{F}(X \times_Y X) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \dots .$$

   is a limit diagram in An *for any $f\colon X \to Y$ in S.*

3. *A sheaf* $\mathcal{F}\colon \mathcal{C}^{op} \to \mathrm{An}$ *is a hypersheaf*[4] *with respect to the Grothendieck pretopology if and only if it more generally takes any* semi-simplicial S-hypercover *in* $\mathcal{C}$*, i.e. an augmented semi-simplicial diagram* $X_\bullet \to Y$ *such that the* n*-th matching map*

$$X_n \to M_n(X) \coloneqq \lim_{(\Delta^{inj}_{\leq n-1})_{/[n]})^{op}} X_k \qquad \textit{(computed in } \mathcal{C}_{/Y}\textit{)}$$

is in S *for all* $n \geq 0$*, to a limit diagram in* An.

Remark 2.2. In particular, the composite $j(-)^{\#} : \mathcal{C} \to \mathbf{Sh}(\mathcal{C})$ preserves finite coproducts. This allows us to rephrase Proposition 1.6, namely $\alpha\colon \mathcal{F} \to \mathcal{G}$ is surjective if and only if, for any $U \in \mathcal{C}$ and map $j(U)^{\#} \to \mathcal{G}$, there exists a morphism $U' \to U$ in S and a commutative diagram

$$\begin{array}{ccc} j(U')^{\#} & \dashrightarrow & \mathcal{F} \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ j(U)^{\#} & \longrightarrow & \mathcal{G}. \end{array}$$

The composite $j(-)^{\#}$ preserves pullbacks and S is stable under pullbacks, so in particular this implies that $j(-)^{\#}$ takes any morphism in S to a surjection in $\mathbf{Sh}(\mathcal{C})$.

Proof. The claim 1. is straightforward. To prove 2., let $\mathcal{F}\colon \mathcal{C}^{op} \to \mathrm{An}$ be a presheaf. In the case that $\mathcal{F}$ is a sheaf with respect to this Grothendieck pretopology, it preserves finite products by applying the sheaf condition to the cover $(U_i \to \bigsqcup_{i=1}^n U_i)_{i=1,\dots,n}$ and noting that the resulting simplicial diagram is constant since coproducts are disjoint, and it satisfies Čech descent by applying the sheaf condition to a cover consisting of a single morphism in S. In the converse direction, assume that $\mathcal{F}$ preserves finite products and satisfies Čech descent. For any cover $(U_i \to U)_{i\in I}$ with I a finite set and $f\colon \coprod_{i=1}^n U_i \to U$ the corresponding morphism in S, we obtain a commutative triangle

$$\begin{array}{ccc} \coprod_{i=1}^n j(U_i) & \longrightarrow & j(\coprod_{i=1}^n U_i) \\ & \searrow \quad \swarrow_{j(f)} & \\ & j(U) & \end{array} \tag{1}$$

and hence a natural transformation $\check{C}(\mathfrak{U}) \Rightarrow \check{C}(j(f))$ of simplicial objects in $\mathbf{PSh}(\mathcal{C})_{/j(U)}$, whose $[n]$-th component is the coproduct-comparison map $\coprod_{i_0,\dots,i_n\in I} j(U_{i_0\dots i_n}) \to j(\coprod_{i_0,\dots,i_n\in I} U_{i_0\dots i_n})$ and thus $\mathcal{F}$ is local w.r.t. this map as it preserves finite products. The maps w.r.t. which $\mathcal{F}$ is local are closed under colimits, so we obtain that $\mathcal{F}$ is local w.r.t. the geometric realization $|\check{C}(\mathfrak{U})| \to |\check{C}(j(f))|$. Furthermore, $\mathcal{F}$ is local w.r.t. $|\check{C}(j(f))| \hookrightarrow j(U)$ as this is precisely Čech descent w.r.t. f, so $\mathcal{F}$ is local w.r.t. to the composite $|\check{C}(\mathfrak{U})| \to j(U)$ (induced by (1)), as desired.

To prove 3., in one direction note that the functor $j(-)^{\#} : \mathcal{C} \to \mathbf{Sh}(\mathcal{C})$ takes semi-simplicial S-hypercovers to semi-simplicial hypercovers in the sense of [Lur18, Definition A.5.2.3] by the preceding remark, so that any hypersheaf satisfies descent for these by [Lur18, Corollary A.5.3.3]. In the converse direction, let $\mathcal{F}$ be a sheaf

[4] i.e. a hypercomplete object in $\mathbf{Sh}(\mathcal{C}, \tau)$.

satisfying this condition. There is an ∞-connective map $\alpha\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a hypersheaf, by general theory. It suffices to show that α is an equivalence, i.e. that $\mathcal{F}(X) \to \mathcal{G}(X)$ is an equivalence for all $X \in \mathcal{C}$. The ∞-topos An is hypercomplete, so it suffices to prove that $\mathcal{F}(X) \to \mathcal{G}(X)$ is ∞-connective for all $X \in \mathcal{C}$. The hypotheses pass to diagonals, so by induction using [Lur09, Proposition 6.5.1.18], it suffices to prove that $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective for all $X \in \mathcal{C}$. To this end, let $j(X)^{\#} \to \mathcal{G}$ be an arbitrary map and it is our goal to construct a semi-simplicial S-hypercover $X_\bullet \to X$ in $\mathcal{C}$ and a commutative square

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \\ \uparrow & & \uparrow \\ j(X_\bullet)^{\#} & \longrightarrow & j(X)^{\#} \end{array}$$

by induction over the skeleta $(\Delta^{inj}_{\leq n})^{op}$, $n \geq 0$. The base case $n = 0$ is doable by Proposition 1.6. In the inductive step, we obtain a diagram

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\Delta^{n+1}\alpha} & M_{n+1}(\mathrm{const}_{\mathcal{F}}) & \rightrightarrows & \mathcal{F} \\ \uparrow & & \uparrow & & \uparrow \\ j(X_{n+1})^{\#} & \longrightarrow & M_{n+1}(j(X_\bullet)^{\#}) & \rightrightarrows & j(X_n)^{\#} \end{array} \qquad \begin{array}{ccc} \xrightarrow{\alpha} & \mathcal{G} \\ & \uparrow \\ \longrightarrow & j(X)^{\#}. \end{array}$$

Indeed, the $(n+1)$-th matching object of the constant diagram at $\mathcal{F}$ is given by the powering $\mathcal{F} \to \mathcal{F}^{S^n}$ (in $\mathbf{Sh}(\mathcal{C})_{/\mathcal{G}}$)[(5)], which is equivalent to the $(n+1)$-fold iterated diagonal of α by a straightforward induction. The functor $j(-)^{\#}$ preserves finite limits, so $M_{n+1}(j(X_\bullet)^{\#}) \simeq j(M_{n+1}(X_\bullet))^{\#}$ is representable. Thus, there is a morphism $X_{n+1} \to M_{n+1}(X_\bullet)$ in S and the diagram can be completed in the lower left corner as indicated by another application of Proposition 1.6 ($\Delta^{n+1}\alpha$ is surjective by [Lur09, Proposition 6.5.1.18]). This describes the extension of $X_\bullet$ over $(\Delta^{inj}_{\leq n+1})^{op}$ by [Lur09, Remark A.2.9.16]. Thus, the induction goes through and constructs an element of $\mathcal{F}(X) \simeq \lim_{\Delta^{inj}} \mathcal{F}(X_\bullet)$ lifting the given element in $\mathcal{G}(X)$ (up to equivalence). □

The presence of semi-simplicial rather than simplicial diagrams in the above may be considered a deficiency. In fact, the argument can be adopted to simplicial hypercovers under a very mild additional assumption.

Proposition 2.3. *In the setting of Proposition 2.1, assume additionally that S satisfies the property that $(f, \mathrm{id}_Y)\colon X \sqcup Y \to Y$ is in S for all $f\colon X \to Y$. Then, a sheaf $\mathcal{F}\colon \mathcal{C}^{op} \to$ An is a hypersheaf with respect to the Grothendieck pretopology if and only if it takes any S-*hypercover *in $\mathcal{C}$, i.e. an augmented simplicial diagram $X_\bullet \to Y$ such that the n-th matching map*

$$X_n \to M_n(X) \coloneqq \lim_{(\Delta^{inj}_{\leq n-1})_{/[n]})^{op}} X_k \qquad \textit{(computed in } \mathcal{C}_{/Y}\textit{)}$$

is in S for all $n \geq 0$, to a limit diagram in An.

[(5)] This is because the $(n+1)$-th matching category $(\Delta^{inj}_{\leq n})_{/[n+1]})^{op}$ is a poset isomorphic to the barycentric subdivision of $\partial\Delta^{n+1}$, hence has geometric realization S^n.

Proof. In one direction, note that the functor $j(-)^{\#} : \mathcal{C} \to \mathbf{Sh}(\mathcal{C})$ takes S-hypercovers to hypercovers in the sense of [Lur09, Definition 6.5.3.2] as before, so that any hypersheaf satisfies descent for these by [Lur09, Lemma 6.5.3.11].

In the converse direction, let $\mathcal{F}$ be a sheaf satisfying this condition. There is an ∞-connective map $\alpha\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a hypersheaf, by general theory. It suffices to show that α is an equivalence, i.e. that $\mathcal{F}(X) \to \mathcal{G}(X)$ is an equivalence for all $X \in \mathcal{C}$. The ∞-topos An is hypercomplete, so it suffices to prove that $\mathcal{F}(X) \to \mathcal{G}(X)$ is ∞-connective for all $X \in \mathcal{C}$. The hypotheses pass to diagonals, so by induction using [Lur09, Proposition 6.5.1.18], it suffices to prove that $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective for all $X \in \mathcal{C}$. To this end, let $j(X)^{\#} \to \mathcal{G}$ be an arbitrary map and it is our goal to construct an S-hypercover $X_\bullet \to X$ in $\mathcal{C}$ and a commutative square

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \\ \uparrow & & \uparrow \\ j(X_\bullet)^{\#} & \longrightarrow & j(X)^{\#} \end{array}$$

by induction over the skeleta $\Delta^{op}_{\leq n}$, $n \geq 0$. Furthermore, we will construct Z_n, $n \geq 0$ in $\mathcal{C}$ such that $X_\bullet|_{(\Delta^{surj})^{op}}$ is left Kan extended from $\{0, 1, \dots\} \to \mathcal{C}$, $k \mapsto Z_k$, i.e. $X_n \simeq \coprod_{[n]\twoheadrightarrow[k]} Z_k$ together with the obvious degeneracy maps. The base case $n = 0$ is doable by Proposition 1.6. In the inductive step, we obtain a diagram

$$\begin{array}{ccccccc} \mathcal{F} & \xrightarrow{\Delta^{n+1}\alpha} & M_{n+1}(\mathcal{F}) & \rightrightarrows & \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ j(Z_{n+1})^{\#} & \longrightarrow & M_{n+1}(j(X_\bullet)^{\#}) & \rightrightarrows & j(X_n)^{\#} & \longrightarrow & j(X)^{\#}, \end{array}$$

where $Z_{n+1} \to M_{n+1}(X_\bullet)$ is a morphism in S by the same argument as in the previous proof. Inductively, $X_\bullet|_{(\Delta^{surj}_{\leq n})^{op}}$ is left Kan extended from $\{0, \dots, n\} \to \mathcal{C}$, $k \mapsto Z_k$, hence the latching diagram

$$(\Delta^{surj}_{\leq n})^{op}_{[n+1]/} \to (\Delta^{surj}_{\leq n})^{op} \to \mathcal{C}$$

is left Kan extended from $\{[n+1] \twoheadrightarrow [k] | k \leq n\} \to \mathcal{C}$, $([n+1] \twoheadrightarrow [k]) \mapsto Z_k$[(6)] and consequently the $(n+1)$-th latching object exists and is given by $L_{n+1}(X_\bullet) \simeq \coprod_{[n+1]\twoheadrightarrow[k],\, k\leq n} Z_k$. The factorization

$$\begin{array}{ccc} & L_{n+1}(X_\bullet) \sqcup Z_{n+1} & \\ \nearrow & & \searrow \\ L_{n+1}(X_\bullet) & \longrightarrow & M_{n+1}(X_\bullet) \end{array}$$

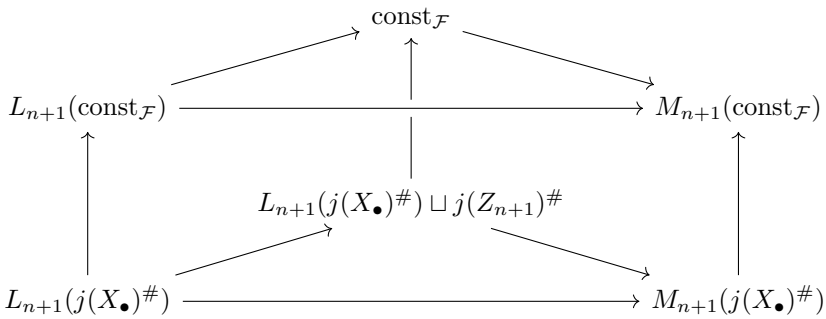
then describes the extension of $X_\bullet$ over $\Delta^{op}_{\leq n+1}$ by [Lur09, Remark A.2.9.16]. Furthermore, the down-right map factors as

$$L_{n+1}(X_\bullet) \sqcup Z_{n+1} \to L_{n+1}(X_\bullet) \sqcup M_{n+1}(X_\bullet) \to M_{n+1}(X_\bullet).$$

The first morphism is in S by stability under finite coproducts and the second morphism is in S by our extra

(6) This can be justified e.g. by [Cis19, Theorem 6.4.13].

assumption, hence the composite is contained in S as well. This guarantees the S-hypercover condition. Lastly, note that the latching object is preserved by $j\,(-)^{\#}$ (as it is a finite coproduct), hence we obtain a commutative diagram



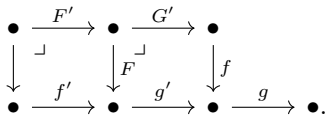
that describes the extension of $j(X_\bullet)^{\#} \to \mathcal{F}$ over $\Delta^{op}_{\leq n+1}$ (as well as morphism to $j(X)^{\#} \to \mathcal{G}$, though the resulting diagram is 4-dimensional). Thus, the induction goes through and constructs an element of $\mathcal{F}(X) \simeq \lim_{\Delta} \mathcal{F}(X_\bullet)$ lifting the given element in $\mathcal{G}(X)$ (up to equivalence). □

Remark 2.4. I do not know whether simplicial hypercovers are sufficient to characterize hypersheaves in absence of the extra assumption on S. A similar condition has also been developed in [Man22, Definition A.3.12., Proposition A.3.16]. Furthermore, the extra condition on S is harmless, as the next proposition shows.

Proposition 2.5. *In the setting of Proposition 2.1, let* $S' \coloneqq \{g | \exists f : gf \in S\}$. *Then,* S' *satisfies the assumptions of Proposition 2.1 and a presheaf* $\mathcal{F} \colon \mathcal{C}^{op} \to \mathrm{An}$ *is a (hyper-)sheaf with respect to the Grothendieck pretopology induced by* S *if and only if it is a (hyper-)sheaf with respect to the Grothendieck pretopology induced by* S'.

Remark 2.6. The collection S' then clearly satisfies the additional assumption of Proposition 2.3, so we obtain that a presheaf $\mathcal{F} \colon \mathcal{C}^{op} \to \mathrm{An}$ is a hypersheaf with respect to the Grothendieck pretopology induced by S if and only if it preserves finite products and maps any simplicial S'-hypercover in $\mathcal{C}$ to a limit diagram in An.

Proof. It is clear that $S \subseteq S'$, in particular S' contains the equivalences. The stability of S' under pullbacks and finite coproducts is easily seen to be inherited from S. It remains to confirm that S is stable under composition: to this end, let $g, g' \in S'$ be composable and pick f, f' s.t. $gf, g'f' \in S$ and consider the diagram



Then $G'F' \in S$ as it is pulled backed from $g'f' \in S$ and $gf \in S$, hence $(gg')(FF') = (gf)(G'F') \in S$ and $gg' \in S'$. This establishes that S' satisfies the hypotheses of Proposition 2.1.

Thus, a presheaf $\mathcal{F} \colon \mathcal{C}^{op} \to \mathrm{An}$ is a sheaf with respect to the Grothendieck pretopology induced by S resp. S' if and only if it preserves finite products and takes Čech nerves of morphisms in S resp. S' to limit diagrams in An. However, since S resp. S' are stable under pullbacks, the latter condition is the same as asking that every morphism in S resp. S' is 'of universal $\mathcal{F}$-descent' in the sense of [LZ24, Definition 3.3.1], but these conditions are then equivalent by [LZ24, Lemma 3.3.2(3)]. □

3 Sheaves and Slices

In this section, we explain how sheaves behave with respect to slices.

Lemma 3.1. *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{C}_0\colon R$ be a left-exact localization of ∞-categories. This induces a left-exact localization $L'\colon \mathcal{C}_{/c} \rightleftarrows (\mathcal{C}_0)_{/Lc}\colon R'$ for any $c \in \mathcal{C}$. The class of morphisms inverted by L' is precisely the preimage of the class of morphisms inverted by L under the projection $\mathcal{C}_{/c} \to \mathcal{C}$.*

Proof. The map $L'\colon \mathcal{C}_{/c} \to (\mathcal{C}_0)_{/Lc}$ is induced by L on slices, whereas the map $R'\colon (\mathcal{C}_0)_{/Lc} \to \mathcal{C}_{/Lc} \to \mathcal{C}_{/c}$ is induced by R on slices, followed by base-change along the unit map $\eta_c\colon c \to Lc$. This is easily checked to be an adjunction (see e.g. [Lur09, Proposition 5.2.5.1]). For any object $c_0 \to Lc$ in $(\mathcal{C}_0)_{/Lc}$, its image under R' is given by the upper horizontal map in the cartesian square in $\mathcal{C}$

$$\begin{array}{ccc} c' & \longrightarrow & c \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \eta_c} \\ c_0 & \longrightarrow & Lc. \end{array}$$

Since L is left-exact and inverts the right vertical map η_c as $L \dashv R$ is a localization, we deduce that the counit map of $L' \dashv R'$ is an equivalence, i.e. R' is fully faithful and $L' \dashv R'$ is a localization as well. Furthermore, pullbacks in over-slices are computed underlying, hence L' is again left-exact. □

Lemma 3.2. *Let $\mathcal{C}$ be an ∞-category, S a collection of morphisms in $\mathcal{C}$ and c an object of $\mathcal{C}$. Then, letting $\pi\colon \mathcal{C}_{/c} \to \mathcal{C}$ be the projection and $S_{/c} \coloneqq \pi^{-1}(S)$, an object $(f\colon d \to c)$ of $\mathcal{C}_{/c}$ is $S_{/c}$-local if and only if f is right-orthogonal to S in $\mathcal{C}$.*

Proof. Let $(g\colon s \to t) \in S$ be arbitrary and consider the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(t,d) & \xrightarrow{g^\circ} & \mathrm{Map}_{\mathcal{C}}(s,d) \\ \downarrow{\scriptstyle f_\circ} & & \downarrow{\scriptstyle f_\circ} \\ \mathrm{Map}_{\mathcal{C}}(t,c) & \xrightarrow{g^\circ} & \mathrm{Map}_{\mathcal{C}}(s,c). \end{array} \tag{2}$$

Then, f is right-orthogonal to g if and only if this square is cartesian if and only if the induced map on vertical fibers is an equivalence for any basepoint $(t \to c)$, but this map is precisely the pre-composition $\overline{g}^\circ\colon \mathrm{Map}_{\mathcal{C}_{/c}}(t,d) \to \mathrm{Map}_{\mathcal{C}_{/c}}(s,d)$ with the corresponding map $\overline{g} \in S_{/c}$ lying over $g \in S$. □

Corollary 3.3. *In the setting of Lemma 3.2, assume that c is an S-local object of $\mathcal{C}$. Then, the $S_{/c}$-local objects in $\mathcal{C}_{/c}$ are precisely the objects lying over S-local objects in $\mathcal{C}$.*

Proof. In this case, the lower horizontal map in the square (2) is an equivalence, hence the square is cartesian if and only if the upper horizontal map is an equivalence. □

Proposition 3.4. *Let $(\mathcal{C}, \tau)$ be an ∞-site and $\mathcal{F} \in \mathbf{Sh}(\mathcal{C}, \tau)$. There is a canonical equivalence $\mathbf{PSh}(\mathcal{C}_{/\mathcal{F}}) \simeq \mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}$, which restricts to a canonical equivalence $\mathbf{Sh}(\mathcal{C}_{/\mathcal{F}}, \tau') \simeq \mathbf{Sh}(\mathcal{C}, \tau)_{/\mathcal{F}}$. Here, τ' is the Grothendieck topology on $\mathcal{C}_{/\mathcal{F}}$ determined by the property that $\pi\colon \mathcal{C}_{/\mathcal{F}} \to \mathcal{C}$ reflects covering sieves.*

Proof. The Yoneda embedding $\mathcal{C}_{/\mathcal{F}} \hookrightarrow \mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{F}}$ has a canonical extension $\mathbf{PSh}\left(\mathcal{C}_{/\mathcal{F}}\right) \xrightarrow{\sim} \mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{F}}$, which is an equivalence by [Lur09, Corollary 5.1.6.12]. The adjunction $(-)^{\#} \dashv \iota$ passes to slices (e.g. by Lemma 3.1), hence $\mathbf{Sh}\left(\mathcal{C},\tau\right)_{/\mathcal{F}}$ is a left-exact localization of $\mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{F}} \simeq \mathbf{PSh}\left(\mathcal{C}_{/\mathcal{F}}\right)$. Furthermore, if S_0 denotes the class of covering sieves in $\mathbf{PSh}(\mathcal{C})$, then Corollary 3.3 implies that $\mathbf{Sh}(\mathcal{C},\tau)_{/\mathcal{F}}$ identifies with precisely the $(S_0)_{/\mathcal{F}}$-local objects of $\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}$ and the collection $(S_0)_{/\mathcal{F}}$ consists of monomorphisms (since pullbacks in over-slices are computed underlying), so this is a topological localization of $\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}$ and thus identifies with $\mathbf{Sh}\left(\mathcal{C}_{/\mathcal{F}},\tau'\right)$ for a uniquely determined Grothendieck topology τ', by [Lur09, Proposition 6.2.2.17].

The projection $\mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{F}} \to \mathbf{PSh}\left(\mathcal{C}\right)$ is identified with the left Kan extension $\pi_{\#}\colon \mathbf{PSh}\left(\mathcal{C}_{/\mathcal{F}}\right) \to \mathbf{PSh}\left(\mathcal{C}\right)$, so the proof shows that a sieve $U \hookrightarrow j\left(s\right)$ in $\mathcal{C}_{/\mathcal{F}}$ is τ'-covering if and only if $\pi_{\#}U \hookrightarrow \pi_{\#}j\left(s\right) \simeq j\left(\pi s\right)$ sheafifies to an equivalence in $\mathbf{Sh}\left(\mathcal{C},\tau\right)$, i.e. if and only if $\pi_{\#}U \hookrightarrow j\left(\pi s\right)$ is a τ-covering sieve by [Lur09, Lemma 6.2.2.16]. □

Remark 3.5. Using the universal property of $\mathbf{PSh}$, it is easy to check that the inverse equivalence $\mathbf{PSh}\left(C\right)_{/\mathcal{F}} \xrightarrow{\sim} \mathbf{PSh}\left(\mathcal{C}_{/F}\right)$ is given by mapping $\left(\mathcal{G} \to \mathcal{F}\right) \mapsto \left(\left(j\left(c\right) \to \mathcal{F}\right) \mapsto \mathrm{Map}_{\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}}\left(j\left(c\right),\mathcal{G}\right)\right)$.

Remark 3.6. The naturality here deserves careful attention as it can be a source of confusion. A map of (pre-)sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ induces a functor $\alpha_!\colon \mathcal{C}_{/\mathcal{F}} \to \mathcal{C}_{/\mathcal{G}}$ and thus adjoint triples

$$\mathbf{PSh}\left(\mathcal{C}_{/F}\right) \underset{\mathrm{Ran}_{\alpha_!}(-)}{\overset{\mathrm{Lan}_{\alpha_!}(-)}{\underset{\leftarrow (\alpha_!)^{\circ} -}{\rightrightarrows}}} \mathbf{PSh}\left(\mathcal{C}_{/\mathcal{G}}\right), \qquad \mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{F}} \underset{\alpha_*}{\overset{\alpha_!}{\underset{\longleftarrow \alpha^* \longrightarrow}{\rightrightarrows}}} \mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{G}}.$$

These adjoint triples are identified under the given equivalence (this is immediate for the leftmost adjoint and then extends by uniqueness). If $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves, these restrict to adjoint triples

$$\mathbf{Sh}\left(\mathcal{C}_{/F}\right) \underset{\mathrm{Ran}_{\alpha_!}(-)}{\overset{\mathrm{Lan}_{\alpha_!}(-)}{\underset{\leftarrow (\alpha_!)^{*} -}{\rightrightarrows}}} \mathbf{Sh}\left(\mathcal{C}_{/\mathcal{G}}\right), \qquad \mathbf{Sh}\left(\mathcal{C}\right)_{/\mathcal{F}} \underset{\alpha_*}{\overset{\alpha_!}{\underset{\longleftarrow \alpha^* \longrightarrow}{\rightrightarrows}}} \mathbf{Sh}\left(\mathcal{C}\right)_{/\mathcal{G}}.$$

Indeed, the right adjoint triple restricts because sheaves are closed under pullbacks and sheafification is left exact, and it is a consequence that the left adjoint triple then also restricts and they are identified.

This discussion also can be extended to hypersheaves.

Proposition 3.7. *Let $\mathcal{X}$ be an ∞-topos and $X \in \mathcal{X}$. Then, there is a canonical equivalence $\left(\mathcal{X}_{/X}\right)^{\wedge} \simeq \left(\mathcal{X}^{\wedge}\right)_{/X^{\wedge}}$.*

Proof. The ∞-connective maps in $\mathcal{X}_{/X}$ are precisely the maps lying over ∞-connective maps in $\mathcal{X}$ since the projection $\mathcal{X}_{/X} \to \mathcal{X}$ induces an equivalence on over-topoi, hence Lemma 3.1 implies that $\left(\mathcal{X}^{\wedge}\right)_{/X^{\wedge}}$ is the left-exact localization of $\mathcal{X}_{/X}$ at the ∞-connective maps, i.e. the hypercompletion of $\mathcal{X}_{/X}$. □

Corollary 3.8. *Let $(\mathcal{C},\tau)$ be an ∞-site and $\mathcal{F} \in \mathbf{Sh}(\mathcal{C},\tau)$. Then, there is a canonical equivalence $\mathbf{Sh}^{hyp}(\mathcal{C}_{/\mathcal{F}},\tau') \simeq \mathbf{Sh}(\mathcal{C},\tau)^{hyp}_{/\mathcal{F}^{hyp}}$, where τ' is the Grothendieck topology of Proposition 3.4.* □

Finally, we record the generalization to the case where $\mathcal{F}$ is not assumed to be a sheaf. This is a more technical matter, and we record it solely out of independent interest.

Proposition 3.9. *Let $(\mathcal{C},\tau)$ be an ∞-site and $\mathcal{F}\in\mathbf{PSh}(\mathcal{C})$. There is a canonical equivalence $\mathbf{PSh}\left(\mathcal{C}_{/\mathcal{F}}\right)\simeq\mathbf{PSh}\left(\mathcal{C}\right)_{/\mathcal{F}}$, which restricts to a canonical equivalence $\mathbf{Sh}\left(\mathcal{C}_{/\mathcal{F}},\tau'\right)\simeq\mathbf{Sh}\left(\mathcal{C},\tau\right)_{/\mathcal{F}^{\#}}$. Here, τ' is the Grothendieck topology on $\mathcal{C}_{/\mathcal{F}}$ determined by the property that $\pi\colon\mathcal{C}_{/\mathcal{F}}\to\mathcal{C}$ reflects covering sieves.*

Proof. The canonical equivalence $\mathbf{PSh}(\mathcal{C}_{/\mathcal{F}})\xrightarrow{\sim}\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}$ is as in the proof of Proposition 3.4. The adjunction $(-)^{\#}\dashv\iota$ induces a left-exact localization $\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}\rightleftarrows\mathbf{Sh}(\mathcal{C})_{/\mathcal{F}^{\#}}$ by Lemma 3.1. Furthermore, if S_0 denotes the class of covering sieves in $\mathbf{PSh}(\mathcal{C})$, then Lemma 3.2 implies that the $(S_0)_{/\mathcal{F}}$-local objects in $\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}$ are precisely the maps $\mathcal{G}\to\mathcal{F}$ that are right-orthogonal to S_0 in $\mathbf{PSh}(\mathcal{C})$. This is equivalent to being right-orthogonal to the saturated class generated by S_0. Now, since S_0 is closed under base-change, this is the same as the acyclic class generated by S_0 by [ABFJ22, Proposition 3.2.18]. Since S_0 consists of monomorphisms, this is the same as the congruence generated by S_0 by [ABFJ22, Proposition 4.2.12], but this is precisely the class of morphisms inverted by $(-)^{\#}$ (cf. [ABFJ22, Section 4.2.1]). This orthogonality is then equivalent, by [ABFJ22, Proposition 3.1.10], to the square

$$\begin{array}{ccc}\mathcal{G} & \longrightarrow & \mathcal{G}^{\#}\\ \downarrow & & \downarrow\\ \mathcal{F} & \longrightarrow & \mathcal{F}^{\#}\end{array}$$

begin cartesian. This in turn is equivalent to the unit map the above localization at $\mathcal{G}\to\mathcal{F}$ being an equivalence, i.e. that $\mathcal{G}\to\mathcal{F}$ is contained in the essential of image of $\mathbf{Sh}(\mathcal{C})_{/\mathcal{F}^{\#}}$. Thus, we have identified $\mathbf{Sh}(\mathcal{C})_{/\mathcal{F}^{\#}}$ as a topological localization of $\mathbf{PSh}(\mathcal{C})_{/\mathcal{F}}\simeq\mathbf{PSh}(\mathcal{C}_{/\mathcal{F}})$ at the class $(S_0)_{/\mathcal{F}}$ of monomorhisms. The rest of the proof is identical to the proof of Proposition 3.4. □

Corollary 3.10. *Let $(\mathcal{C},\tau)$ be an ∞-site and $\mathcal{F}\in\mathbf{PSh}(\mathcal{C})$. There is a natural equivalence $\mathbf{Sh}^{hyp}(\mathcal{C}_{/\mathcal{F}},\tau')\simeq\mathbf{Sh}(\mathcal{C},\tau)^{hyp}_{/((\mathcal{F}^{\#})^{hyp})}$, where τ' is the Grothendieck topology of Proposition 3.9.* □

4 The Convenient Sites are no Loss of Generality

Finally, we discuss how sheaves on any site that admits pullbacks can be understood in terms of the 'convenient sites' of Proposition 2.1.

Proposition 4.1. *For any small ∞-category $\mathcal{C}$, let $\mathcal{C}^{\sqcup}\subseteq\mathbf{PSh}\left(\mathcal{C}\right)$ denote the full subcategory on the finite coproducts of representables. The Yoneda embedding factors as a fully faithful functor $j\colon\mathcal{C}\hookrightarrow\mathcal{C}^{\sqcup}$ that exhibits the target as the* free finite coproduct-completion *of $\mathcal{C}$, i.e. $\mathcal{C}^{\sqcup}$ admits finite coproducts and, for any ∞-category $\mathcal{D}$ admitting finite coproducts, restriction induces an equivalences*

$$j^{\circ}\colon\ \mathrm{Fun}^{\sqcup}\left(\mathcal{C}^{\sqcup},\mathcal{D}\right)\xrightarrow{\sim}\mathrm{Fun}\left(\mathcal{C},\mathcal{D}\right),$$

where the domain is the full subcategory on all finite coproduct-preserving functors.

Proof. This is a special case of [Lur09, Proposition 5.3.6.2]. □

Lemma 4.2. *For any small ∞-category $\mathcal{C}$, the free finite coproduct-completion $\mathcal{C}^{\sqcup}$ is finitary extensive.*

Proof. The ∞-category $\mathbf{PSh}(\mathcal{C})$ is extensive and the full subcategory $\mathcal{C}^{\sqcup} \subseteq \mathbf{PSh}(\mathcal{C})$ stable under finite coproducts, so it suffices to note that it is also stable under pullbacks along finite coproduct injections in $\mathcal{C}^{\sqcup}$. □

Lemma 4.3. *For any small ∞-category $\mathcal{C}$ that admits pullbacks, the free finite coproduct-completion $\mathcal{C}^{\sqcup}$ also admits pullbacks and the Yoneda embedding $j\colon \mathcal{C} \hookrightarrow \mathcal{C}^{\sqcup}$ preserves them.*

Proof. It suffices to observe that $\mathcal{C}^{\sqcup} \subseteq \mathbf{PSh}(\mathcal{C})$ is stable under pullbacks. This reduces, by universality of coproducts in $\mathbf{PSh}(\mathcal{C})$, to pullbacks of cospans of the form $j(c') \to \sqcup_{i\in I} j(c_i) \leftarrow j(c'')$, but such a pullback is either the initial object $\emptyset$ or computed in $\mathcal{C}$, so the claim follows. □

Proposition 4.4. *For any small ∞-category $\mathcal{C}$ that admits pullbacks with a Grothendieck pretopology ρ whose covers are all finite, the free finite coproduct-completion $j\colon \mathcal{C} \hookrightarrow \mathcal{C}^{\sqcup}$ together with the collection S of morphisms that are finite coproducts of morphisms of the form $\coprod_{i\in I} j(U_i) \to j(U)$ for covers $(U_i \to U)_{i=1,\dots,n}$ in ρ satisfies the hypotheses of Proposition 2.1. Furthermore, for the induced Grothendieck (pre-)topology τ on $\mathcal{C}^{\sqcup}$, restriction induces an equivalence*

$$j^{\circ}\colon \mathbf{Sh}(\mathcal{C}^{\sqcup}, \tau) \xrightarrow{\sim} \mathbf{Sh}(\mathcal{C}, \rho)\,.$$

Proof. The preceding lemmas imply that $\mathcal{C}^{\sqcup}$ admits pullbacks as well as finite coproducts that are universal and disjoint. The collection S clearly contains the equivalences and is stable under finite coproducts. The universality of finite coproducts in $\mathcal{C}^{\sqcup}$ and the fact that j preserves pullbacks reduce the stability of S under pullbacks and composition to the corresponding properties of ρ in $\mathcal{C}$. To verify the last claim, consider the diagram

$$\begin{array}{ccccc} \mathbf{Sh}(\mathcal{C}^{\sqcup}, \tau) & \subseteq & \mathrm{Fun}^{\sqcap}(\mathcal{C}^{\sqcup,op}, \mathrm{An}) & \subseteq & \mathrm{Fun}(\mathcal{C}^{\sqcup,op}, \mathrm{An}) \\ & \searrow & & \overset{\sim}{\searrow} & \downarrow{\scriptstyle j^{\circ}} \\ & & \mathbf{Sh}(\mathcal{C}, \rho) & \subseteq & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{An})\,. \end{array}$$

The indicated equivalence holds by the universal property of the free finite coproduct-completion. Thus, it remains, by Proposition 2.1, to prove that a finite product-preserving functor $\mathcal{F}\colon \mathcal{C}^{\sqcup,op} \to \mathrm{An}$ takes Čech nerves of morphisms in S to limit diagrams in An if and only if $\mathcal{F}j\colon \mathcal{C}^{op} \to \mathrm{An}$ is a ρ-sheaf. To this end, note that universality and disjointness of finite coproducts imply that taking Čech nerves commutes with finite coproducts, so finite product-preservation reduces the former condition to those morphisms in S of the form $\coprod_{i\in I} j(U_i) \to U$ for a ρ-cover $(U_i \to U)_{i\in I}$, but $\mathcal{F}$ takes the Čech nerve of this morphism to a limit diagram if and only if $\mathcal{F}j$ satisfies the sheaf condition for that cover, because $\mathcal{C}^{\sqcup} \hookrightarrow \mathbf{PSh}(\mathcal{C})$ preserves finite coproducts and pullbacks. □

Remark 4.5. This entire discussion could be repeated with the word 'finite' replaced by 'κ-small' throughout, for any regular cardinal κ. In particular, the result could then be applied to *any* small ∞-site that admits pullbacks by picking a large enough regular cardinal κ.

References

[ABFJ22] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. "Left-exact localizations of ∞-topoi I: Higher sheaves". In: *Advances in Mathematics* 400, 108268 (2022).

[Bas26] Connor Bass. *Geometric Categories and Sheaves on Topoi*. 2026. arXiv: 2605.02115 [math.CT]. URL: https://arxiv.org/abs/2605.02115.

[Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019.

[Gei25] Thorger Geiß. *Light Condensed Anima and Étale Sheaves*. 2025. URL: https://thorgergeiss.gitlab.io/Light_Condensed_Anima.pdf.

[Hoy26] Marc Hoyois. *Lectures on Higher Topos Theory*. Lecture Notes by Bastiaan Cnossen. 2026. URL: https://bastiaancnossen.com/notes/topos-theory.pdf.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009.

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry*. 2018. URL: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[LZ24] Yifeng Liu and Weizhe Zheng. *Enhanced six operations and base change theorem for higher Artin stacks*. 2024. arXiv: 1211.5948 [math.AG]. URL: https://arxiv.org/abs/1211.5948.

[Man22] Lucas Mann. "A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry". PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, Aug. 2022. URL: https://hdl.handle.net/20.500.11811/10125.