AN EXAMPLE IN HIGHER TOPOS THEORY

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We give examples of topoi[(1)] that are not Postnikov-complete, yet in which every Postnikov tower converges. These examples were discovered in a conversation with Phil Pützstück, and are partially inspired by [CM21, Example 2.18]. The question of whether such topoi exist was previously raised in [Hai22, Warning 1.23], [AB24, Remark A.8] or also [NP26, Example 2.29].

Recall that in any presentable category $\mathcal{C}$, the full subcategory $\tau_{\leq n}\mathcal{C}$ on the n-truncated objects for any $n \geq -1$ is an accessible localization of $\mathcal{C}$ [Lur09, Proposition 5.5.6.18]. This begets a tower in Pr^L

$$\mathcal{C} \longrightarrow \ldots \longrightarrow \tau_{\leq n}\mathcal{C} \longrightarrow \ldots \longrightarrow \tau_{\leq 0}\mathcal{C} \longrightarrow \tau_{\leq -1}\mathcal{C}.$$

Taking the limit yields a canonical left adjoint $\iota\colon \mathcal{C} \to \hat{\mathcal{C}} \coloneqq \lim_n \tau_{\leq n}\mathcal{C}$, whose right adjoint is given by $(X_n)_n \mapsto \lim_n X_n$ (cf. [Lur18, Remark A.7.3.6]). The unit of this adjunction evaluated an object X is the canonical map $X \to \lim_n \tau_{\leq n}X$ comparing X to the limit of its Postnikov tower. We say that *Postnikov towers in $\mathcal{C}$ converge* if this map is an equivalence for all $X \in \mathcal{C}$, equivalently the canonical map $\iota\colon \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ is fully faithful. We say that $\mathcal{C}$ is *Postnikov-complete* if the canonical map $\iota\colon \mathcal{C} \xrightarrow{\sim} \hat{\mathcal{C}}$ is an equivalence.

Proposition 1. *Let $\mathcal{X}$ be a non-hypercomplete topos such that the full subcategory $\mathcal{X}^\wedge \subseteq \mathcal{X}$ on the hypercomplete objects is stable under small colimits. Fix a left-exact accessible localization $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{X}$ for some small category $\mathcal{C}$. Then the pullback $\mathcal{T} \coloneqq \mathrm{PSh}(\mathcal{C}) \times_{\mathcal{X}} \mathcal{X}^\wedge$ is a topos that is not Postnikov-complete, yet in which every Postnikov tower converges.*

Proof. The category $\mathcal{T}$ is defined by the pullback

$$\begin{array}{ccc} \mathcal{T} & \hookrightarrow & \mathrm{PSh}(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle L} \\ \mathcal{X}^\wedge & \hookrightarrow & \mathcal{X}. \end{array}$$

The bottom horizontal functor is always a right adjoint, and it is a left adjoint by the assumption and the adjoint functor theorem. In particular, it is an algebraic morphism of topoi. The functor L is also an algebraic morphism of topoi, so this entire pullback lives in LTop and $\mathcal{T}$ is a topos by [Lur09, Proposition 6.3.2.3].

For any $n \geq 0$, the left-exact functor $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{X}$ preserves n-truncated objects [Lur09, Proposition 5.5.6.16], and n-truncated objects are hypercomplete [Lur09, Lemma 6.5.2.9], hence $\mathcal{T}$ contains all the n-truncated objects of $\mathrm{PSh}(\mathcal{C})$, which are a fortiori n-truncated in $\mathcal{T}$. In the converse direction, the inclusion $\mathcal{T} \subseteq \mathrm{PSh}(\mathcal{C})$ is also left-exact, hence n-truncated objects in $\mathcal{T}$ are also n-truncated in $\mathrm{PSh}(\mathcal{C})$. In total, this implies that $\tau_{\leq n}\mathcal{T} = \tau_{\leq n}\mathrm{PSh}(\mathcal{C})$ and the truncation functors on $\mathrm{PSh}(\mathcal{C})$ restrict to those on $\mathcal{T}$.

This implies that the inclusion $\mathcal{T} \subseteq \mathrm{PSh}(\mathcal{C})$ induces an equivalence on Postnikov-completions, but any presheaf topos is Postnikov-complete, hence the inclusion exhibits $\mathrm{PSh}(\mathcal{C})$ as the Postnikov-completion of $\mathcal{T}$. As L is essentially surjective and $\mathcal{X}$ is not hypercomplete, this inclusion is proper and thus $\mathcal{T}$ is not Postnikov-complete. However, Postnikov towers in $\mathcal{T}$ agree with those in $\mathrm{PSh}(\mathcal{C})$ by the previous considerations and are thus a posteriori convergent in $\mathcal{T}$ as well. □

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(1)The phrase 'category' resp. 'topos' always means '∞-category' resp. '∞-topos'.

Example 2. To produce an example of a topos to which this proposition applies, we use topoi of parametrized objects as introduced by Joyal and studied by Hoyois [Hoy19] (see also [Hoy26, Section 6.11]). Let $\mathcal{C}$ be a presentable stable category. The topos of An-parametrized objects of $\mathcal{C}$ is the cartesian unstraightening $\pi\colon \int_{\mathrm{An}} \mathcal{C} \to \mathrm{An}$ of the functor $\mathrm{An}^{\mathrm{op}} \to \widehat{\mathrm{Cat}},\ X \mapsto \mathrm{Fun}(X, \mathcal{C})$. In this case, π admits a fully faithful two-sided adjoint $\mathrm{An} \hookrightarrow \int_{\mathrm{An}} \mathcal{C},\ X \mapsto (X, 0)$ (e.g. by using the theory of relative adjunctions [Lur17, Proposition 7.3.2.6], as π is a bicartesian fibration). We claim that its essential image consists precisely of the hypercomplete objects in the topos $\int_{\mathrm{An}} \mathcal{C}$, hence they are stable under small colimits.

To see this, we note that the geometric morphism $\pi\colon \int_{\mathrm{An}} \mathcal{C} \to \mathrm{An}$ preserves ∞-connective maps, hence these map to equivalences in An, i.e. are given fiber-wise. The fiber inclusion $\mathrm{Fun}(X, \mathcal{C}) \to \int_{\mathrm{An}} \mathcal{C}$, for any $X \in \mathrm{An}$, preserves weakly contractible (co)limits and thus ∞-connective maps. In a stable category, all maps are ∞-connective, so we conclude that ∞-connective maps in $\int_{\mathrm{An}} \mathcal{C}$ are precisely the fiber-wise maps, from which the conclusion is easily deduced by orthogonality.

Example 3. In the case $\mathcal{C} = \mathrm{Sp}$ is the category of spectra, one can identify the topos $\int_{\mathrm{An}} \mathrm{Sp} \simeq \mathrm{Exc}^1(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ with the category of (non-reduced) 1-excisive functors $\mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$, which arises via a left-exact accessible localization $P_1\colon \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \to \mathrm{Exc}^1(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$, the first *Goodwillie approximation* [Lur17, Theorem 6.1.1.10 & Remark 6.1.1.11]. The hypercomplete objects correspond precisely to the constant functors, so the topos $\mathcal{T}$ in this case is the category of functors $F\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$ whose first Goodwillie approximation $P_1 F$ is a constant functor.

For completeness' sake, let us note that the left adjoint P_1 can be described as a relative left adjoint [Lur17, Proposition 7.3.2.6], which is given fiber-wise over $X \in \mathrm{An}$ (using evaluation) by the Goodwillie approximation $\mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}_{X/\!\!/X}) \to \mathrm{Exc}^1_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}_{X/\!\!/X})$ for *reduced* functors with target a pointed category. This admits the explicit formula $F \mapsto \mathrm{colim}_{n \geq 0}\, \Omega^n_X F(\Sigma^n(-))$ (note that Σ is the usual suspension, but Ω_X is the loop functor as calculated in $\mathrm{An}_{X/\!\!/X}$ rather than An). In particular, it is immediate from this description that P_1 maps truncated presheaves to constant functors, which is all that is necessary for the above argument.

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