A NOTE ON THE FUNCTORIALITY OF THE SHAPE OF ∞-TOPOI

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The *shape* of an (∞-)topos $\mathcal{X}$ is a fundamental invariant $\Pi_\infty(\mathcal{X}) \in \mathrm{Pro}(\mathrm{An})$, introduced in [TV03] and studied further in [Lur09, Section 7.1.6]. The shape defines a functor $\Pi_\infty \colon \mathrm{RTop} \to \mathrm{Pro}(\mathrm{An})$, cf. [Lur09, Remark 7.1.6.15], [Hoy18, p. 4] and, especially, [Lur18, Proposition E.2.2.1].

There are also a number of ad hoc constructions of the shape as a functor that appear in the literature, in particular also in the author's previous work, and their equivalence is only understood implicitly. The purpose of this note is to address that deficiency, which we will do in the generality of the *relative shape*.

Theorem A. *Let $\mathcal{Y}$ be a topos. The following functors $\Pi^{\mathcal{Y}}_\infty(-) \colon \mathrm{RTop}_{/\mathcal{Y}} \to \mathrm{Pro}(\mathcal{Y})$ are equivalent:*

1. *The pro-left adjoint of the left-exact, accessible functor $\mathcal{Y}_{/(-)} \colon \mathcal{Y} \to \mathrm{RTop}_{/\mathcal{Y}}$, $Y \mapsto (Y_* \colon \mathcal{Y}_{/Y} \to \mathcal{Y})$ (the original definition, cf. [Lur18, Proposition E.2.2.1], [CE18, Section 4.1]).*
2. *The functor $\mathrm{RTop}_{/\mathcal{Y}} \to \mathrm{Pro}(\mathcal{Y})$, $(f_* \colon \mathcal{X} \to \mathcal{Y}) \mapsto \mathcal{X}_* f^*$, where the functoriality is obtained by mate calculus, cf. Definition 1.2 (a similar definition appears in forthcoming work of Lehner–Volpe).*
3. *The functor $\mathrm{RTop}_{/\mathcal{Y}} \to \mathrm{Pro}(\mathcal{Y})$, $(f_* \colon \mathcal{X} \to \mathcal{Y}) \mapsto \mathcal{X}_* f^*$, where the functoriality is obtained by the unit maps of the adjunctions, cf. Definition 2.4 (the definition implicit in [Gei26, Proposition C]).*

These functors are called the relative shape.

Furthermore, the constructions (2)–(3) in Theorem A are naturally upgraded to (equivalent) 2-functors.

Theorem B. *There is a canonical enhancement of the relative shape to a 2-functor $\Pi^{\mathcal{Y}}_\infty \colon \mathbf{RTop}_{/\mathcal{Y}} \to \mathrm{Pro}(\mathcal{Y})$.*

Remark. *The target $\mathrm{Pro}(\mathcal{Y})$ is still a 1-category, i.e. this 2-functor canonically inverts all 2-morphisms in the source. This confirms a prediction made in [Leh26, Remark 3.10].*

Conventions. We will make use of the following conventions:

- A morphism of topoi is denoted $f^* \colon \mathcal{X} \rightleftarrows \mathcal{Y} \colon f_*$, where f^* is the *algebraic* and f_* the *geometric* morphism. For any topos $\mathcal{X}$, there exists a unique morphism of topoi $\mathcal{X}^* \colon \mathrm{An} \rightleftarrows \mathcal{X} \colon \mathcal{X}_*$.
- If $p \colon \mathcal{E} \to \mathcal{C}$ is a cocartesian resp. cartesian fibration of (∞-)categories and $f \colon x \to y$ a morphism in $\mathcal{C}$, then we denote by $f_\# \colon \mathcal{E}_x \to \mathcal{E}_y$ resp. $f^* \colon \mathcal{E}_y \to \mathcal{E}_x$ the cocartesian resp. cartesian transport.
- For generic 1- resp. 2-categories, we use caligraphic resp. boldface caligraphic letters.
- The notation for (op)lax slices we use is that of [AHM26, Definition 2.5.1].
- The cocartesian/$(0,1)$-unstraightening of a forgetful functor $U \colon \mathcal{C} \to 2\mathrm{Cat}$ is denoted $p \colon \widetilde{\mathcal{C}} \to \mathcal{C}$.

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1. Theorem A, (1)–(2) and Theorem B

First, we explain the functor (1) of Theorem A.

Proposition 1.1 ([Lur18, Proposition E.2.2.1],[CE18, Section 4.1]). *The pro-extension $\beta^{\mathcal{Y}} \colon \mathrm{Pro}(\mathcal{Y}) \to \mathrm{RTop}_{\mathcal{Y}/}$ of $\mathcal{Y}_{/(-)} \colon \mathcal{Y} \to \mathrm{RTop}_{/\mathcal{Y}}$, $Y \mapsto (Y_* \colon \mathcal{Y}_{/Y} \to \mathcal{Y})$ admits a left adjoint $\Pi^{\mathcal{Y}}_\infty(-) \colon \mathrm{RTop}_{/\mathcal{Y}} \to \mathrm{Pro}(\mathcal{Y})$.*

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Proof. For a geometric morphism of topoi $f_*\colon \mathcal{X} \to \mathcal{Y}$, we have $\mathrm{Map}_{\mathrm{RTop}_{/\mathcal{Y}}}(\mathcal{X}, \mathcal{Y}_{/Y}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, f^*(Y))$, naturally in $Y \in \mathcal{Y}$ ([Lur09, Corollary 6.3.5.6], or [Hoy26, Proposition 4.38]). In particular, since $\mathcal{X}$ is an accessible category and f^* is a geometric morphism, we see that $\Pi^{\mathcal{Y}}_\infty(\mathcal{X}) \coloneqq \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, f^*(-))$ is a pro-anima with the property that $\mathrm{Map}_{\mathrm{Pro(An)}}(\Pi^{\mathcal{Y}}_\infty(\mathcal{X}), Y) \simeq \mathrm{Map}_{\mathrm{RTop}_{/\mathcal{Y}}}(\mathcal{X}, \beta^{\mathcal{Y}}(Y))$, naturally in $Y \in \mathcal{Y}$. Then, pro-extending this equivalence, we have verified the local criterion for the existence of an adjunction. □

Now, using mate calculus, we construct functor (2) of Theorem A as a 2-functor, proving Theorem B along the way. To this end, we recall that here exists a canonical equivalence $(\mathbf{Cat}^R)^{\mathrm{coop}} \simeq \mathbf{Cat}^L$ (e.g. [HM24, Theorem D.3.17] or [HHLN23, Theorem B]), restricting to $\mathbf{RTop}^{\mathrm{coop}} \simeq \mathbf{LTop}$.

Definition 1.2. For any topos $\mathcal{Y}$, we consider the composite 2-functor

$$\mathbf{LTop}_{\mathcal{Y}/} \longrightarrow \mathbf{Cat}_{\mathcal{Y}/} \times_{\mathbf{Cat}} \mathbf{Cat}_{\nearrow \mathcal{Y}} \xrightarrow{\circ} \mathrm{Hom}_{\widehat{\mathbf{Cat}}}(\mathcal{Y}, \mathcal{Y}) \simeq \mathrm{Fun}(\mathcal{Y}, \mathcal{Y}) \xrightarrow{\mathcal{Y}_*} \mathrm{Fun}(\mathcal{Y}, \mathrm{An}).$$

Here, the first functor is induced by the composite $\mathbf{LTop}_{\mathcal{Y}/} \hookrightarrow \mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{Y}\swarrow} \simeq \mathbf{Cat}^{\mathrm{radj}}_{\nearrow \mathcal{Y}} \hookrightarrow \mathbf{Cat}_{\nearrow \mathcal{Y}}$ (the superscripts $(-)^{\mathrm{ladj}}$ resp. $(-)^{\mathrm{radj}}$ denote the full subcategories on those objects whose structure maps are left resp. right adjoints) and the inclusion $\mathbf{LTop}_{\mathcal{Y}/} \hookrightarrow \mathbf{Cat}_{\mathcal{Y}/}$.

This maps $(f^*\colon \mathcal{Y} \to \mathcal{X}) \mapsto \mathcal{Y}_* f_* f^* \simeq \mathcal{X}_* f^*$ on objects, which is a left-exact, accessible functor. Thus, the composite factors through $\mathrm{Fun}^{\mathrm{lex,acc}}(\mathcal{Y}, \mathrm{An}) \simeq \mathrm{Pro}(\mathcal{Y})^{\mathrm{op}}$ and applying $(-)^{\mathrm{coop}}$ yields a 2-functor

$$\mathbf{RTop}_{/\mathcal{Y}} \simeq (\mathbf{LTop}_{\mathcal{Y}/})^{\mathrm{coop}} \longrightarrow \mathrm{Pro}(\mathcal{Y})^{\mathrm{co}} \simeq \mathrm{Pro}(\mathcal{Y}).$$

Proposition 1.3. *The pro-left adjoint of Proposition 1.1 is equivalent to the underlying* 1*-functor of Definition 1.2.*

Proof Sketch. To this end, recall the functor $\mathcal{Y}_{/(-)}\colon \mathcal{Y}^{\mathrm{op}} \to \mathbf{LTop}_{\mathcal{Y}/}$, $Y \mapsto (Y \times (-)\colon \mathcal{Y} \to \mathcal{Y}_{/Y})$. It fits into a commutative diagram

$$\begin{array}{ccccccc} \mathcal{Y}^{\mathrm{op}} & \longrightarrow & \mathbf{Cat}^{\mathrm{ladj,radj}}_{\mathcal{Y}/} \times_{\mathbf{Cat}} \mathbf{Cat}^{\mathrm{radj}}_{\nearrow \mathcal{Y}} & \xrightarrow{\circ} & \mathrm{Fun}^R(\mathcal{Y}, \mathcal{Y}) & \xrightarrow{\mathcal{Y}_*} & \mathrm{Fun}^R(\mathcal{Y}, \mathrm{An}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{LTop}_{\mathcal{Y}/} & \longrightarrow & \mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{Y}/} \times_{\mathbf{Cat}} \mathbf{Cat}^{\mathrm{radj}}_{\nearrow \mathcal{Y}} & \xrightarrow{\circ} & \mathrm{Fun}(\mathcal{Y}, \mathcal{Y}) & \xrightarrow{\mathcal{Y}_*} & \mathrm{Fun}(\mathcal{Y}, \mathrm{An}). \end{array} \tag{1}$$

The full subcategory inclusion $\mathrm{Fun}^R(\mathcal{Y}, \mathrm{An}) \subseteq \mathrm{Fun}^{\mathrm{lex,acc}}(\mathcal{Y}, \mathrm{An})$ identifies with the opposite of the Yoneda embedding $\mathcal{Y}^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(\mathcal{Y})^{\mathrm{op}}$, and we claim that this is precisely the total composite in (1). To begin, the naturality of [HHLN23, Theorem 5.3.6] (cf. [AGH24, Theorem 5.3.8]) yields a commutative square

$$\begin{array}{ccc} \mathbf{Cat}^{\mathrm{ladj,radj}}_{\mathcal{Y}/} \times_{\mathbf{Cat}} \mathbf{Cat}^{\mathrm{radj}}_{\nearrow \mathcal{Y}} & \xrightarrow{\circ} & \mathrm{Fun}^R(\mathcal{Y}, \mathcal{Y}) \\ \downarrow & & \downarrow \wr \\ \mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{Y}\swarrow} \times_{\mathbf{Cat}} \mathbf{Cat}^{\mathrm{ladj}}_{\swarrow \mathcal{Y}} & \xrightarrow{\circ} & \mathrm{Fun}^L(\mathcal{Y}, \mathcal{Y})^{\mathrm{op}}, \end{array}$$

where the vertical maps are passing to adjoints. The composite $\mathcal{Y}^{\mathrm{op}} \to \mathrm{Fun}^R(\mathcal{Y}, \mathcal{Y}) \simeq \mathrm{Fun}^L(\mathcal{Y}, \mathcal{Y})^{\mathrm{op}}$ thus identifies with the composite (where U denotes the forgetful functor)

$$\mathcal{Y}^{\mathrm{op}} \to \mathrm{Fun}^L(\mathcal{Y}, \mathcal{Y})^{\mathrm{op}},\ Y \mapsto (Y \times (-)\colon \mathcal{Y} \to \mathcal{Y}_{/Y}, U\colon \mathcal{Y}_{/Y} \to \mathcal{Y}) \mapsto (Y \times (-)\colon \mathcal{Y} \to \mathcal{Y}).$$

Then passing back to right adjoints, we see that the total composite in the diagram (1) is given by the functor $\mathcal{Y}^{\mathrm{op}} \to \mathbf{LTop}_{\mathcal{Y}/} \to \mathrm{Fun}(\mathcal{Y}, \mathrm{An})$, $Y \mapsto \mathrm{Map}_{\mathcal{Y}}(Y, -)$. Then applying $(-)^{\mathrm{op}}$ and recalling the construction of Pro, the composite $\Pi^{\mathcal{Y}}_\infty \circ \mathcal{Y}_{/(-)}\colon \mathcal{Y} \to \mathrm{Pro}(\mathcal{Y})$ is precisely the Yoneda embedding, i.e. $\Pi^{\mathcal{Y}}_\infty \circ \mathcal{Y}_{/(-)} \xrightarrow{\sim} j$.

Using the universal property of right Kan extension, we can extend this natural equivalence along the Yoneda embedding to a natural transformation $\Pi^{\mathcal{Y}}_{\infty} \circ \beta^{\mathcal{Y}} \Rightarrow \mathrm{id}_{\mathrm{Pro}(\mathcal{Y})}$. We claim that this is a counit transformation exhibiting $\Pi^{\mathcal{Y}}_{\infty} \dashv \beta^{\mathcal{Y}}$ as an adjunction. For this, we need to check that for any topos $f_*\colon \mathcal{X} \to \mathcal{Y}$ over $\mathcal{Y}$ and pro-object $Y_\bullet \in \mathrm{Pro}(\mathcal{Y})$, the following composite is an equivalence:

$$\mathrm{Map}_{\mathrm{RTop}_{/\mathcal{Y}}}(\mathcal{X}, \beta^{\mathcal{Y}}(Y_\bullet)) \xrightarrow{\Pi^{\mathcal{Y}}_{\infty}} \mathrm{Map}_{\mathrm{Pro}(\mathcal{Y})}(\Pi^{\mathcal{Y}}_{\infty}(\mathcal{X}), \Pi^{\mathcal{Y}}_{\infty}\beta^{\mathcal{Y}}(Y_\bullet)) \xrightarrow{\varepsilon} \mathrm{Map}_{\mathrm{Pro}(\mathcal{Y})}(\Pi^{\mathcal{Y}}_{\infty}(\mathcal{X}), Y_\bullet).$$

The source and target are compatible with cofiltered limits in $Y_\bullet$, so it suffices to check this for elements in the essential image of the Yoneda embedding $\mathcal{Y} \hookrightarrow \mathrm{Pro}(\mathcal{Y})$, where it unwinds to the equivalence of [Lur09, Corollary 6.3.5.6] (or [Hoy26, Proposition 4.38]). □

2. Theorem A, (3)

To define functor (3) of Theorem A, we first review a general 'push-pull' construction.

Proposition 2.1. *Let $p\colon \mathbf{E} \to \mathbf{B}$ be both a $(0,1)$- and a $(1,1)$-fibration of 2-categories. Suppose that $0 \in \mathbf{B}$ is 2-initial, i.e. $\mathrm{Hom}_{\mathbf{B}}(0,b) \simeq \{\varphi_b\}$ for all $b \in \mathbf{B}$. Then there exists a canonical 2-functor*

$$\rho_p\colon \mathbf{B} \to \mathbf{Fun}(\mathbf{E}_0, \mathbf{E}_0),\ b \mapsto (\varphi_b)^*(\varphi_b)_\#.$$

This is natural w.r.t. pullback along functors preserving the 2-initial object.

Proof. The $(0,1)$-straightening [AGH24, Theorem 3.5.4] of p is denoted $\mathrm{Str}^{(0,1)}(p)\colon \mathbf{B} \to 2\mathbf{Cat}$. The identity $\mathrm{id}\colon \mathbf{E}_0 \xrightarrow{\sim} \mathbf{E}_0$ corresponds to a natural transformation $\mathrm{const}_{\mathbf{E}_0} \Rightarrow \mathrm{Str}^{(0,1)}(p)$ (since the limit of a $\mathbf{B}$-indexed diagram is given by evaluation at the initial object $0 \in \mathbf{B}$) and thus under the straightening-unstraightening equivalence to a map of $(0,1)$-fibrations $\varphi_\#\colon \mathbf{B} \times \mathbf{E}_0 \to \mathbf{E}$, $(b,e) \mapsto (\varphi_b)_\#(e)$ over $\mathbf{B}$.

On the other hand, considering the $(1,1)$-straightening $\mathrm{Str}^{(1,1)}(p)\colon \mathbf{B}^{\mathrm{coop}} \to 2\mathbf{Cat}$ of p instead, the identity $\mathrm{id}\colon \mathbf{E}_0 \to \mathbf{E}_0$ corresponds to a natural transformation $\mathrm{Str}^{(1,1)}(p) \Rightarrow \mathrm{const}_{\mathbf{E}_0}$ and thus under the straightening-unstraightening equivalence to a map of $(1,1)$-fibrations $\varphi^*\colon \mathbf{E} \to \mathbf{B} \times \mathbf{E}_0$, $e \mapsto (\varphi_{p(e)})^*(e)$ over $\mathbf{B}$.

The composite

$$\mathbf{B} \times \mathbf{E}_0 \xrightarrow{\varphi_\#} \mathbf{E} \xrightarrow{\varphi^*} \mathbf{B} \times \mathbf{E}_0 \xrightarrow{\mathrm{pr}} \mathbf{E}_0$$

then curries to the desired functor $\rho\colon \mathbf{B} \to \mathbf{Fun}(\mathbf{E}_0, \mathbf{E}_0)$. □

Remark 2.2. This construction applies in particular to any bicartesian fibration of 1-categories. Furthermore, if $p\colon \mathbf{E} \to \mathbf{B}$ is as in the proposition, then the underlying functor $p_{\leq 1}\colon \mathbf{E}_{\leq 1} \to \mathbf{B}_{\leq 1}$ of 1-categories is a bicartesian fibration, and the construction is compatible in the sense that the following triangle commutes:

$$\begin{array}{ccc} \mathbf{B}_{\leq 1} & \xrightarrow{(\rho_p)_{\leq 1}} & \mathbf{Fun}(\mathbf{E}_0, \mathbf{E}_0)_{\leq 1} \\ & \searrow^{\rho_{p_{\leq 1}}} & \downarrow \\ & & \mathrm{Fun}((\mathbf{E}_0)_{\leq 1}, (\mathbf{E}_0)_{\leq 1}). \end{array}$$

Note that the vertical comparison map is an equivalence if p is already 1-fibred.

Example 2.3. For a 2-category $\mathbf{C}$, we consider the the forgetful functor $2\mathbf{Cat}^L_{\mathbf{C}/} \to 2\mathbf{Cat}^L \hookrightarrow 2\mathbf{Cat}$ and its $(0,1)$-unstraightening $\widetilde{2\mathbf{Cat}^L_{\mathbf{C}/}} \to 2\mathbf{Cat}^L_{\mathbf{C}/}$, which is also a $(1,1)$-fibration by [AGH24, Corollary 5.1.6]. The 2-category $2\mathbf{Cat}^L_{\mathbf{C}/}$ admits the 2-initial object $\mathrm{id}_{\mathbf{C}}$, so Proposition 2.1 induces a functor

$$\rho_{\mathbf{C}}\colon 2\mathbf{Cat}^L_{\mathbf{C}/} \to \mathbf{Fun}(\mathbf{C}, \mathbf{C}),\ (F\colon \mathbf{C} \to \mathbf{D}) \mapsto F^R F.$$

This example is, in a certain sense, universal. Indeed, let $p\colon \mathbf{E} \to \mathbf{B}$ be as in Proposition 2.1 and $\mathrm{Str}^{(0,1)}(p)\colon \mathbf{B} \to 2\mathbf{Cat}^L$ its $(0,1)$-straightening. Then we can lift $\mathrm{Str}^{(0,1)}(p)$ through a functor $\mathbf{B} \simeq \mathbf{B}_{0/} \to 2\mathbf{Cat}^L_{\mathbf{E}_0/}$ by slicing

over 0 and naturality implies that ρ_p identifies with the composite

$$\mathbf{B} \longrightarrow 2\mathbf{Cat}^L_{\mathrm{E}_0/} \xrightarrow{\rho_{\mathrm{C}}} \mathbf{Fun}(\mathbf{E}_0, \mathbf{E}_0).$$

Now, we can state Definition (3) of the relative shape.

Definition 2.4. For any topos $\mathcal{Y}$, we consider the composite 2-functor

$$\mathbf{LTop}_{\mathcal{Y}/} \hookrightarrow \mathbf{Cat}^L_{\mathcal{Y}/} \xrightarrow{\rho_{\mathcal{Y}}} \mathrm{Fun}(\mathcal{Y}, \mathcal{Y}) \xrightarrow{\mathcal{Y}_*} \mathrm{Fun}(\mathcal{Y}, \mathrm{An}).$$

This maps $(f^*\colon \mathcal{Y} \to \mathcal{X}) \mapsto \mathcal{Y}_* f_* f^* \simeq \mathcal{X}_* f^*$ on objects, which is a left-exact, accessible functor. Thus, the composite factors through $\mathrm{Fun}^{\mathrm{lex,acc}}(\mathcal{Y}, \mathrm{An}) \simeq \mathrm{Pro}(\mathcal{Y})^{\mathrm{op}}$ and applying $(-)^{\mathrm{coop}}$ yields a 2-functor

$$\mathbf{RTop}_{/\mathcal{Y}} \simeq (\mathbf{LTop}_{\mathcal{Y}/})^{\mathrm{coop}} \longrightarrow \mathrm{Pro}(\mathcal{Y})^{\mathrm{co}} \simeq \mathrm{Pro}(\mathcal{Y}).$$

Remark 2.5. The underlying 1-functor is, by Remark 2.2, the one obtained by the same construction using the bicartesian fibration $\widetilde{\mathrm{Cat}^L_{\mathcal{Y}/}} \to \mathrm{Cat}^L_{\mathcal{Y}/}$ of 1-categories instead.

To compare (2) and (3), we identify the relevant (co-)cartesian transports.

Lemma 2.6. *For any category $\mathcal{C}$, the composite $\mathcal{C} \times \mathbf{Cat}_{\mathcal{C}/} \to \widetilde{\mathbf{Cat}_{\mathcal{C}/}} \to \mathbf{Cat}_{*/\nearrow}$ of the cocartesian transport $\varphi_\#$ for the 2-initial object $\mathrm{id}_{\mathcal{C}} \in \mathbf{Cat}^L_{\mathcal{C}/}$ and the projection is given by the map $(X, F\colon \mathcal{C} \to \mathcal{D}) \mapsto (\mathcal{D}, F(X))$.*

Proof. The commutative square

$$\begin{array}{ccc} \mathcal{C} \times \mathbf{Cat}_{\mathcal{C}/} & \xrightarrow{\circ} & \mathbf{Cat}_{*/\nearrow} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\mathcal{C}/} & \longrightarrow & \mathbf{Cat} \end{array}$$

induces a map $\mathcal{C} \times \mathbf{Cat}_{\mathcal{C}/} \to \widetilde{\mathbf{Cat}_{\mathcal{C}/}} \simeq \mathbf{Cat}_{\mathcal{C}/} \times_{\mathbf{Cat}} \mathbf{Cat}_{*/\nearrow}$ over $\mathbf{Cat}_{\mathcal{C}/}$, and it suffices to identify this map with the cocartesian transport for the 2-initial object $\mathrm{id}_{\mathcal{C}} \in \mathbf{Cat}^L_{\mathcal{C}/}$. First, we check that this is a map of (1-fibred) $(0,1)$-fibrations. Indeed, a cocartesian morphism $(X, F\colon \mathcal{C} \to \mathcal{D}) \to (G(X), F'\colon \mathcal{C} \to \mathcal{D}')$ lying over a map $G\colon \mathcal{D} \to \mathcal{D}'$ under $\mathcal{C}$ is mapped to $(\mathcal{D}, F(X)) \to (\mathcal{D}', F'(X))$, which is a cocartesian morphism for $\mathbf{Cat}_{*/\nearrow} \to \mathbf{Cat}$ as the structure map $GF(X) \xrightarrow{\sim} F'(X)$ is an equivalence. Thus, as in the proof of Proposition 2.1, this map of $(0,1)$-fibrations is uniquely determined by its restriction to the fiber $\mathcal{C} \simeq \widetilde{\mathbf{Cat}_{\mathcal{C}/}} \times_{\mathbf{Cat}_{\mathcal{C}/}} \{\mathrm{id}_{\mathcal{C}}\}$, which is by construction the identity $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$. □

Lemma 2.7. *For any category $\mathcal{C}$, the composite $\widetilde{\mathbf{Cat}^L_{\mathcal{C}/}} \hookrightarrow \widetilde{\mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{C}/}} \simeq \widetilde{\mathbf{Cat}^{\mathrm{radj}}_{/\mathcal{C}}} \xrightarrow{\circ} \mathcal{C}$ induces a map of (1-fibred) $(1,1)$-fibrations $\widetilde{\mathbf{Cat}^L_{\mathcal{C}/}} \to \mathbf{Cat}^L_{\mathcal{C}/} \times \mathcal{C}$ over $\mathbf{Cat}^L_{\mathcal{C}/}$, which is equivalent to the cartesian transport φ^* for the 2-initial object $\mathrm{id}_{\mathcal{C}} \in \mathbf{Cat}^L_{\mathcal{C}/}$.*

Proof. First of all, $\mathbf{Cat}^L_{\mathcal{C}/} \to \mathbf{Cat}^L \hookrightarrow \mathbf{Cat}$ takes values in left adjoints by construction, hence the $(0,1)$-fibration $\widetilde{\mathbf{Cat}^L_{\mathcal{C}/}} \to \mathbf{Cat}^L_{\mathcal{C}/}$ is also a $(1,1)$-fibration by [AGH24, Corollary 5.1.6]. For a left adjoint $F\colon (L\colon \mathcal{C} \to \mathcal{D}) \to (L'\colon \mathcal{C} \to \mathcal{D}')$ under $\mathcal{C}$ and a lift $(\mathcal{C} \to \mathcal{D}', d') \in \widetilde{\mathbf{Cat}^L_{\mathcal{C}/}}$, its cartesian transport is the morphism $(\mathcal{C} \to \mathcal{D}, f^R(d')) \to (\mathcal{C} \to \mathcal{D}', d')$ given by the underlying left adjoint F and the counit $\varepsilon\colon FF^R(d') \to d'$ in $\mathcal{D}'$. For the first part, it suffices to show that these cartesian morphisms are inverted, but its image is

$$RF^R(d') \longrightarrow R'FF^R(d') \xrightarrow{R'\varepsilon} R'(d'),$$

which is an equivalence by mate calculus. It then follows, as in the proof of Proposition 2.1, that this map of $(1,1)$-fibrations is uniquely determined by its restriction to the fiber $\mathcal{C} \simeq \widetilde{\mathbf{Cat}^L_{\mathcal{C}/}} \times_{\mathbf{Cat}^L_{\mathcal{C}/}} \{\mathrm{id}_{\mathcal{C}}\}$, which is by construction the identity $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$. □

Proposition 2.8. *The 2-functors of Definitions 1.2 and 2.4 are equivalent.*

Proof. It suffices to prove the two respective 2-functors $\mathbf{Cat}^L_{\mathcal{C}/} \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$ (cf. Definition 1.2) are equivalent. To this end, we consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{Cat}^L_{\mathcal{C}/} \times \mathcal{C} & \hookrightarrow & \mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{C}/} \times \mathcal{C} & \longrightarrow & (\mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{C}/} \times_{\mathbf{Cat}} \mathbf{Cat}^{\mathrm{radj}}_{\nearrow \mathcal{C}}) \times \mathcal{C} & \longrightarrow & \mathrm{Fun}(\mathcal{C},\mathcal{C}) \times \mathcal{C}. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\
\widetilde{\mathbf{Cat}^L_{\mathcal{C}/}} & \hookrightarrow & \widetilde{\mathbf{Cat}^{\mathrm{ladj}}_{\mathcal{C}/}} & \hookrightarrow & \mathbf{Cat}_{*\nearrow} \times_{\mathbf{Cat}} \mathbf{Cat}^{\mathrm{radj}}_{\nearrow \mathcal{C}} & \xrightarrow{\circ} & \mathcal{C}.
\end{array}$$

The right-down composite curries to the 2-functor $\mathbf{Cat}^L_{\mathcal{C}/} \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$ of Definition 1.2, which induces the relative shape functor of Definition 1.2. The vertical functors in the left square are given by cocartesian transport, hence it commutes trivially. The middle square commutes by the identification of cocartesian transport with composition of Lemma 2.6, and the right square commutes by associativity of composition. To conclude, we observe that the down-right composite is identified with the 2-functor $\mathbf{Cat}^L_{\mathcal{C}/} \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$ of Definition 2.4 by Lemma 2.7. □

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