FUN WITH TOPOLOGICAL K-THEORY SPECTRA

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This note concerns the (connective) topological K-theory spectra ku resp. ko, as well as their non-connective versions KU resp. KO. These are $\mathbb{E}_\infty$-ring spectra and we take the computation of their homotopy groups, i.e. Bott periodicity, for granted. Armed therewith, we present a number of computations: we prove Wood's theorem, calculate the $\mathbb{F}_2$-homology of ku resp. ko and calculate the Tate L-theory spectra $L^{\mathrm{t}}(\mathrm{ku})$ resp. $L^{\mathrm{t}}(\mathrm{ko})$. The latter involves, in particular, a calculation of the spectra ku^{hC_2} and ku^{tC_2} resp. ko^{hC_2} and ko^{tC_2} (w.r.t. the trivial C_2-actions), for which we give an elementary argument. In an appendix, we also compute the $\mathbb{E}_\infty$-ring endomorphisms of these K-theory spectra.

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1. Recollections on K-theory Spectra

Let KO resp. KU denote the (non-connective) real resp. complex topological K-theory spectru and ko resp. ku their connective covers. These are $\mathbb{E}_\infty$-ring spectra and their graded homotopy rings are given by $\pi_*(\mathrm{KU}) = \mathbb{Z}[\beta^{\pm 1}]$ resp. $\pi_*(\mathrm{ku}) = \mathbb{Z}[\beta]$, where $|\beta| = 2$, and

$$\pi_*(\mathrm{KO}) = \mathbb{Z}[\eta, \alpha, \beta_{\mathbb{R}}^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta_{\mathbb{R}}) \qquad \text{resp.} \qquad \pi_*(\mathrm{ko}) = \mathbb{Z}[\eta, \alpha, \beta_{\mathbb{R}}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta_{\mathbb{R}}),$$

where $|\eta| = 1$, $|\alpha| = 4$ and $|\beta_{\mathbb{R}}| = 8$. There is a canonical $\mathbb{E}_\infty$-ring map $c\colon \mathrm{KO} \to \mathrm{KU}$, called *complexification*, by which we regard KU as a KO-algebra. Furthermore, C_2 acts KO-linearly on KU via complex conjugation (i.e. the Adams operation Ψ^{-1}) and this induces an equivalence $\mathrm{KO} \xrightarrow{\sim} \mathrm{KU}^{hC_2}$ resp. $\mathrm{ko} \xrightarrow{\sim} \tau_{\geq 0}\mathrm{ku}^{hC_2}$ (cf. [HS14, Section 5]).

2. Wood's Theorem & Realification

In this section, we prove Wood's Theorem. The proof presented is based on the one in [Mat15, Section 3].

Lemma 2.1. *For any finite spectrum X, the natural map $\mathrm{KO} \otimes X \to (\mathrm{KU} \otimes X)^{hC_2}$ is an equivalence of* KO*-modules.*

Proof. This is a natural transformation of exact functors in X, which is an equivalence for $X \simeq \mathbb{S}$. □

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Proposition 2.2. *The complex K-theory of $\mathbb{CP}^2$ is given by $\mathrm{KU}^0(\mathbb{CP}^2) \cong \mathbb{Z}[t]/(t^3)$ as a ring, where $t = [L]-1$ is the virtual difference of the tautological line bundle and the trivial rank 1 bundle. The induced C_2-action by complex conjugation is determined by $t \mapsto -t + t^2$, so in particular $\widetilde{\mathrm{KU}}^0(\mathbb{CP}^2) \cong \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{Z})$ as C_2-representations.*

Proof. The (multiplicative) AHSS immediately collapses and yields that $\mathrm{KU}^0(\mathbb{CP}^2) \cong \mathbb{Z}[t]/(t^3)$, where t can be chosen to be any element of $\mathrm{KU}^0(\mathbb{CP}^2)$ sitting in filtration degree 2 and mapping to a generator of $E_\infty^{2,-2} \cong \mathbb{Z}$. Now, note that $F^2\mathrm{KU}^0(\mathbb{CP}^2) = \widetilde{\mathrm{KU}}^0(\mathbb{CP}^2)$ and the projection $F^2\mathrm{KU}^0(\mathbb{CP}^2) \to E_\infty^{2,-2}$ identifies by naturality with the restriction map $\widetilde{\mathrm{KU}}^0(\mathbb{CP}^2) \to \widetilde{\mathrm{KU}}^0(\mathbb{CP}^1)$, which takes $[L]-1$ to a generator by the geometric formulation of Bott periodicity.

To identify the C_2-action, recall that $L \otimes \overline{L} \cong 1$ (as for any line bundle), hence the C_2-action takes $1+t \mapsto \frac{1}{1+t} = 1 - t + t^2$ and thus $t \mapsto -t + t^2$. In particular, the C_2-translates of t form an additive basis of $\tilde{\mathrm{KU}}^0(\mathbb{CP}^2)$, so this C_2-representation is induced. □

To avoid notational ambiguity, let $\mathrm{KU}^{\heartsuit}$ denote the $\mathbb{E}_\infty$-algebra in the category Sp^{BC_2} of spectra with a C_2-action given by KU with the Adams operation Ψ^{-1}. The category of 'C_2-equivariant $\mathrm{KU}^{\heartsuit}$-modules' $\mathrm{Mod}_{\mathrm{KU}^{\heartsuit}}(\mathrm{Sp}^{BC_2})$ consists of ordinary KU-modules with a C_2-action that is linear over Ψ^{-1}, and there is a free-forgetful adjunction

$$\mathrm{KU}^{\heartsuit} \otimes (-) \colon \mathrm{Sp}^{BC_2} \rightleftarrows \mathrm{Mod}_{\mathrm{KU}}(\mathrm{Sp}^{BC_2}).$$

In particular, $\mathrm{KU}^{\heartsuit}$ itself represents the functor $\Omega^\infty(-)^{hC_2} \colon \mathrm{Mod}_{\mathrm{KU}}(\mathrm{Sp}^{BC_2}) \to \mathrm{An}$, i.e. homotopy classes of $\mathrm{KU}^{\heartsuit}$-linear maps $\mathrm{KU}^{\heartsuit} \to M$ are in bijection with $\pi_0(M)^{C_2}$.

Proposition 2.3. *There is an equivalence $\mathrm{KU}^{\heartsuit} \otimes \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{S}) \xrightarrow{\sim} \mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}$ of $\mathrm{KU}^{\heartsuit}$-modules under $\mathrm{KU}^{\heartsuit}$.*

Proof. First, let's consider the underlying KU-modules. There, $\mathrm{KU}^{\heartsuit} \otimes \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{S})$ is underlying $\mathrm{KU} \otimes (\mathbb{S} \oplus \mathbb{S}) \simeq \mathrm{KU} \oplus \mathrm{KU}$ on the one hand, and there is a splitting $\mathrm{KU} \otimes \mathbb{S}/\eta \simeq \mathrm{KU} \oplus \Sigma^2\mathrm{KU} \simeq \mathrm{KU} \oplus \mathrm{KU}$ as $\eta \colon \Sigma\mathrm{KU} \to \mathrm{KU}$ is null, because it acts KU-linearly via the image of η in $\pi_1(\mathrm{KU}) = 0$, where we have also used Bott periodicity.

Now, we compute that $\pi_0(\mathrm{KU} \otimes \mathbb{S}/\eta) \cong \widetilde{\mathrm{KU}}_2(\mathbb{CP}^2) \cong \widetilde{\mathrm{KU}}^2(\mathbb{CP}^2) \cong \widetilde{\mathrm{KU}}^0(\mathbb{CP}^2)$, where we have used $\mathbb{S}/\eta \simeq \Sigma^{\infty-2}\mathbb{CP}^2$, the UCT (these K-groups are finite free) and, finally, Bott periodicity. The isomorphisms except for the last are C_2-equivariant, whereas Bott periodicity twists with the sign representation. This implies by Proposition 2.2 that $\pi_0(\mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}) \cong \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{Z}) \otimes \mathbb{Z}^\sigma \cong \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{Z})$ as C_2-representations. Now, a choice of $x \in \pi_0(\mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}})$ whose C_2-translates form a basis induces a C_2-equivariant map $\mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{S}) \to \mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}$ and hence a $\mathrm{KU}^{\heartsuit}$-linear map $\mathrm{KU}^{\heartsuit} \otimes \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{S}) \to \mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}$, which is underlying at the level of KU-modules a π_0-isomorphism and hence an equivalence, since source and target are both free KU-modules of rank 2.

Finally, we need to choose x so that this equivalence is a map under $\mathrm{KU}^{\heartsuit}$ via the canonical structure maps. The composite $\mathrm{KU}^{\heartsuit} \to \mathrm{KU}^{\heartsuit} \otimes \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{S}) \to \mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}$ is classified by the element $x + \Psi^{-1}.x \in \pi_0(\mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}})^{C_2}$, as one checks by unwinding the construction. The canonical map $\mathrm{KU}^{\heartsuit} \to \mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}$, on the other hand, is classified by the image of the generator under $\mathbb{Z} \cong \widetilde{\mathrm{KU}}_2(S^2) \to \widetilde{\mathrm{KU}}_2(\mathbb{CP}^2)$, which, by naturality of the previous identifications, is the element dual to $\beta^{-1}t$. An easy computation with the action dual to the one obtained in Proposition 2.2 (twisted by a sign) shows that letting x be minus the dual of $\beta^{-1}t^2$ does the job. □

Remark 2.4. The map $\eta \colon \Sigma\mathrm{KU} \to \mathrm{KU}$ is null, but not C_2-equivariantly null. Indeed, if that were the case, the splitting $\mathrm{KU} \otimes \mathbb{S}/\eta \simeq \mathrm{KU} \oplus \Sigma^2\mathrm{KU}$ would be C_2-equivariant, hence $\pi_0(\mathrm{KU} \otimes \mathbb{S}/\eta) = \mathbb{Z} \oplus \mathbb{Z}$ with the action $(x, y) \mapsto (x, -y)$, which is not an induced representation, contradicting our calculations.

Theorem 2.5 (Wood). *There is a canonical exact sequence of KO-modules*

$$\Sigma\mathrm{KO} \xrightarrow{\eta} \mathrm{KO} \xrightarrow{c} \mathrm{KU}.$$

Proof. The $\mathrm{KU}^{\heartsuit}$-linear equivalence $\mathrm{KU}^{\heartsuit} \otimes \mathrm{Ind}_{\{e\}}^{C_2}(\mathbb{S}) \xrightarrow{\sim} \mathrm{KU}^{\heartsuit} \otimes (\mathbb{S}/\eta)^{\mathrm{triv}}$ under $\mathrm{KU}^{\heartsuit}$ of Proposition 2.3 induces, by taking homotopy C_2-fixed points, a KO-linear equivalence $\mathrm{KU} \xrightarrow{\sim} \mathrm{KO} \otimes \mathbb{S}/\eta$ under KO, using Lemma 2. □

Corollary 2.6. *There is a canonical exact sequence*

$$\Sigma \mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \xrightarrow{c} \mathrm{ku}.$$

Proof. It suffices to exhibit a ko-linear equivalence $\mathrm{ko} \otimes \mathbb{S}/\eta \xrightarrow{\sim} \mathrm{ku}$ under ko. This is a consequence of Wood's Theorem since the map $\mathrm{ko} \otimes \mathbb{S}/\eta \to \tau_{\geq 0}(\mathrm{KO} \otimes \mathbb{S}/\eta)$ is an equivalence: it suffices to observe that $\tau_{\leq -1}\mathrm{KO} \otimes \mathbb{S}/\eta \simeq \mathrm{fib}(\Sigma\eta \colon \Sigma^2\tau_{\leq -1}\mathrm{KO} \to \Sigma\tau_{\leq -1}\mathrm{KO})$ is (-1)-coconnective since $\tau_{\leq -1}\mathrm{KO}$ is already (-3)-coconnective (in fact, (-4)-coconnective). □

Corollary 2.7. *The ring map $c \colon \mathrm{KO} \to \mathrm{KU}$ is a C_2-Galois extension.*

Proof. As we have already recalled, C_2 acts KO-linearly on KU via the Adams operation Ψ^{-1} and this induces an equivalence $\mathrm{KO} \xrightarrow{\sim} \mathrm{KU}^{hC_2}$. We can then use the Wood cofiber sequence of Theorem 2.5 to calculate

$$\mathrm{KU} \otimes_{\mathrm{KO}} \mathrm{KU} \simeq \mathrm{cofib}(\Sigma\mathrm{KU} \xrightarrow{\eta} \mathrm{KU}) \simeq \mathrm{KU} \oplus \Sigma^2\mathrm{KU} \simeq \mathrm{KU} \oplus \mathrm{KU}$$

as KO-modules, where we have used that η acts trivially on KU by evenness, as well as Bott periodicity. □

Finally, we will use the Wood cofiber sequence to define the realification map.

Definition 2.8. The *realification* is the KO-linear map $r \colon \mathrm{KU} \to \mathrm{KO}$ given by

$$\mathrm{KU} \xrightarrow{\beta} \Sigma^{-2}\mathrm{KU} \xrightarrow{\partial} \mathrm{KO},$$

where ∂ is the boundary map in the Wood cofiber sequence, resp. its connective cover $r \colon \mathrm{ku} \to \mathrm{ko}$.

Proposition 2.9. *The complexification and realification maps are related by $rc \simeq 2$ and $cr \simeq 1 + \Psi^{-1}$ up to a (consistent) sign.*

Remark 2.10. In particular, we see that $r(1) = 2$, so that r cannot be refined to an $\mathbb{E}_\infty$-ring map.

Proof. The map $rc \colon \mathrm{KO} \to \mathrm{KO}$ is KO-linear, hence determined by its action on $1 \in \pi_0(\mathrm{KO})$. Let's consider the long exact sequence of homotopy groups

$$\pi_2(\mathrm{KO}) \xrightarrow{c} \pi_2(\mathrm{KU}) \xrightarrow{\partial} \pi_0(\mathrm{KO}) \xrightarrow{\eta} \pi_1(\mathrm{KO}).$$

The left-hand map is null because it has the type $\mathbb{Z}/2 \to \mathbb{Z}$ and the right-hand map is the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$, hence we see that $\partial(\beta) = \pm 2$, as desired.

For the second claim, we first note that the Wood cofiber sequence induces an exact sequence

$$\mathrm{Hom}_{\mathrm{KO}}(\Sigma^2\mathrm{KO}, \mathrm{KU}) \longrightarrow \mathrm{Hom}_{\mathrm{KO}}(\mathrm{KU}, \mathrm{KU}) \longrightarrow \mathrm{Hom}_{\mathrm{KO}}(\mathrm{KO}, \mathrm{KU}) \quad \simeq \quad \mathrm{KU}.$$

The latter map is given by restriction along $c \colon \mathrm{KO} \to \mathrm{KU}$ and evaluating at $1 \in \pi_0(\mathrm{KO})$, i.e. by evaluating at $1 \in \pi_0(\mathrm{KU})$. The associated long exact sequence on homotopy groups splits by evenness into short exact sequences, in particular we obtain a short exact sequence $0 \to \mathbb{Z} \to \pi_0 \mathrm{Hom}_{\mathrm{KO}}(\mathrm{KU}, \mathrm{KU}) \to \mathbb{Z} \to 0$, which is necessarily split.

Note that any KO-linear map $\mathrm{KU} \to \mathrm{KU}$ in the image of the first map factors as $\mathrm{KU} \to \Sigma^2\mathrm{KO} \to \mathrm{KU}$, where the first map is the boundary map ∂ and the second map classifies an element $n\beta \in \pi_2(\mathrm{KU})$, so that the composite map acts as $\beta \mapsto \pm 2n\beta$ on $\pi_2(\mathrm{KU})$ by the previous computation. The KO-linear map $1 - \Psi^{-1} \colon \mathrm{KU} \to \mathrm{KU}$, which annihilates $1 \in \pi_0(\mathrm{KU})$ and hence lies in the essential image of the first map, acts as $\beta \mapsto 2\beta$, hence $1 - \Psi^{-1}$ generates this summand. The identity is an obvious section, so we ultimately obtain a splitting $\pi_0 \mathrm{Hom}_{\mathrm{KO}}(\mathrm{KU}, \mathrm{KU}) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{\Psi^{-1}\}$.

This implies every KO-linear map $\mathrm{KU} \to \mathrm{KU}$ is of the form $x + y\Psi^{-1}$ for some integers $x, y \in \mathbb{Z}$. This map acts on π_0 resp. π_2 via $1 \mapsto x + y$ resp. $\beta \mapsto (x - y)\beta$, so this action uniquely determines the map. For the composite cr, we have $cr(1) = 2$ by the previous computation and $cr(\beta) = 0$ as $\pi_2(\mathrm{KO})$ is torsion and $\pi_2(\mathrm{KU})$ is torsion-free, hence in total $cr \simeq 1 + \Psi^{-1}$. □

From this, we can completely describe the actions of r and c on homotopy groups (note both maps are KO-linear and hence act 8-periodically):

$$c(\eta) = 0, \qquad c(\alpha) = 2\beta^2, \qquad c(\beta_{\mathbb{R}}) = \beta^4,$$
$$r(1) = 2, \qquad r(\beta) = \eta^2, \qquad r(\beta^2) = \alpha, \qquad r(\beta^3) = 0.$$

3. $\mathbb{F}_2$-homology of ku and ko

In this section, we compute the $\mathbb{F}_2$-homology and induced homomorphisms of the maps

$$\mathrm{ko} \xrightarrow{c} \mathrm{ku} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{F}_2.$$

To do so, we recall that $H_*(\mathbb{F}_2;\mathbb{F}_2) \cong \mathcal{A}_* \cong \mathbb{F}_2[\zeta_1,\zeta_2,\dots]$ is the dual mod 2 Steenrod algebra, which is polynomial on the indicated generators in degrees $|\zeta_i| = 2^i - 1$ for $i \geq 0$. We claim that the maps induced by the above sequence are all injective and give the polynomial subalgebras

$$\mathbb{F}_2[\zeta_1^4,\zeta_2^2,\zeta_3,\dots] \hookrightarrow \mathbb{F}_2[\zeta_1^2,\zeta_2^2,\zeta_3,\dots] \hookrightarrow \mathbb{F}_2[\zeta_1^2,\zeta_2,\zeta_3,\dots] \hookrightarrow \mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\dots].$$

For impending use, we also recall the mod 2 Steenrod algebra $H^*(\mathbb{F}_2;\mathbb{F}_2) \cong \mathcal{A}^* \cong \bigoplus_{\mathrm{Sq}^I \,|\, I \text{ admissible}} \mathbb{F}_2$ (this basis is additive, not multiplicative). The bases of $\mathcal{A}^*$ and $\mathcal{A}_*$ are related via the Kronecker pairing

$$\left\langle \mathrm{Sq}^I, \zeta_i \right\rangle = \begin{cases} 1, & I = (2^{i-1},\dots,2,1), \\ 0, & \text{else.} \end{cases}$$

Let us briefly calculate the dual bases in degrees up to 3. In degree 1 resp. 2, the two Steenrod algebras are free on Sq^1 and ζ_1 resp. Sq^2 and ζ_1^2, so those basis elements are necessarily dual to another. In degree 3, the algebras are free on Sq^3 and $\mathrm{Sq}^2\mathrm{Sq}^1$ resp. ζ_2 and ζ_1^3, so there is something to determine. The above formula immediately yields $\left\langle \mathrm{Sq}^3, \zeta_2 \right\rangle = 0$ and $\left\langle \mathrm{Sq}^2\mathrm{Sq}^1, \zeta_2 \right\rangle = 1$. Furthermore, using the explicit formulas for the comultiplication on $\mathcal{A}^*$ resp. $\mathcal{A}_*$ and discarding terms of the wrong degree(s), we obtain

$$\left\langle \mathrm{Sq}^3, \zeta_1^3 \right\rangle = \left\langle \mathrm{Sq}^1 \otimes \mathrm{Sq}^1 \otimes \mathrm{Sq}^1, \zeta_1 \otimes \zeta_1 \otimes \zeta_1 \right\rangle = \left\langle \mathrm{Sq}^1, \zeta_1 \right\rangle^3 = 1,$$
$$\left\langle \mathrm{Sq}^2\mathrm{Sq}^1, \zeta_1^3 \right\rangle = \left\langle \mathrm{Sq}^2 \otimes \mathrm{Sq}^1, \zeta_1^2 \otimes \zeta_1 \right\rangle = \left\langle \mathrm{Sq}^2, \zeta_1^2 \right\rangle \left\langle \mathrm{Sq}^1, \zeta_1 \right\rangle = 1.$$

Thus, Sq^3 is dual to ζ_1^3 and $\mathrm{Sq}^2\mathrm{Sq}^1$ is dual to $\zeta_2 + \zeta_1^3$.

Proposition 3.1. *The* mod 2 *reduction* $\mathbb{Z} \to \mathbb{F}_2$ *induces on* $\mathbb{F}_2$*-homology the inclusion* $\mathbb{F}_2[\zeta_1^2,\zeta_2,\dots] \hookrightarrow \mathbb{F}_2[\zeta_1,\zeta_2,\dots]$.

Proof. To prove this, we consider the exact sequence $\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{F}_2$, where the first map is null upon tensoring with $\mathbb{F}_2$, so the LES of the rotated sequence $\mathbb{Z} \to \mathbb{F}_2 \to \Sigma\mathbb{Z}$ splits into degree-wise SESs. Thus, $\mathbb{Z} \to \mathbb{F}_2$ is injective on $\mathbb{F}_2$-homology and identifies with the kernel of $\mathbb{F}_2 \to \Sigma\mathbb{Z}$ on $\mathbb{F}_2$-homology, but this is then the same as the kernel of the composite $\mathbb{F}_2 \to \Sigma\mathbb{Z} \to \Sigma\mathbb{F}_2$ on $\mathbb{F}_2$-homology. Applying naturality to the diagram of exact sequences

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{F}_2 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{F}_2 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{F}_2 \end{array}$$

identifies this composite with the Steenrod square $\mathrm{Sq}^1 \colon \mathbb{F}_2 \to \Sigma\mathbb{F}_2$, so we obtain an exact sequence

$$0 \longrightarrow H_*(\mathbb{Z};\mathbb{F}_2) \hookrightarrow \mathbb{F}_2[\zeta_1,\zeta_2,\dots] \xrightarrow{(\mathrm{Sq}^1)_*} \mathbb{F}_2[\zeta_1,\zeta_2,\dots][-1].$$

The Steenrod square Sq^1 acts (tautologically) on the Steenrod algebra $\mathcal{A}^*$ by multiplication with Sq^1, so it acts on the dual Steenrod algebra $\mathcal{A}_*$ by the composite

$$\mathcal{A}_* \xrightarrow{\Delta} \mathcal{A}_* \otimes \mathcal{A}_* \xrightarrow{\langle \mathrm{Sq}^1, - \rangle \otimes \mathrm{id}} \mathcal{A}_*,$$

where Δ is the comultiplication. It is then an easy computation that this kills precisely the monomials in which ζ_1 appears to an even power, as desired. □

To calculate the $\mathbb{F}_2$-homologies of ku resp. ko, we proceed analogously. The identification of the relevant boundary maps is done in terms of the first interesting k-invariants of these spectra, and these can be understood in terms of $\mathbb{S}$ resp. $\mathbb{S}/\eta$ in low degrees, whose $\mathbb{F}_2$-homology in turn is understood.

Lemma 3.2. *There exists a* 3*-connected map* $f\colon \mathbb{S}/\eta \to \mathrm{ku}$.

Proof. The unit map $\mathbb{S} \to \mathrm{ko}$ is 3-connected (i.e. has 4-connective cofiber), so tensoring with the connective spectrum $\mathbb{S}/\eta$ yields a 3-connected map $\mathbb{S}/\eta \to \mathrm{ko} \otimes \mathbb{S}/\eta \simeq \mathrm{ku}$ by Wood's Theorem (Corollary 2.6). □

Proposition 3.3. *The truncation* $\mathrm{ku} \to \mathbb{Z}$ *induces on* $\mathbb{F}_2$*-homology the inclusion* $\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \dots] \hookrightarrow \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \dots]$.

Proof. To prove this, we consider the Bott exact sequence $\Sigma^2\mathrm{ku} \to \mathrm{ku} \to \mathbb{Z}$, where the first map is multiplication by the Bott element $\beta \in \pi_2(\mathrm{ku})$. We claim that this map is null upon tensoring with $\mathbb{F}_2$, equivalently that $\mathrm{ku} \to \mathrm{ku} \otimes \mathbb{F}_2$ kills the Bott element β. To this end, let $f\colon \mathbb{S}/\eta \to \mathrm{ku}$ be as in Lemma 3.2 and consider the commutative square

$$\begin{array}{ccc} \mathbb{S}/\eta & \xrightarrow{f} & \mathrm{ku} \\ \downarrow & & \downarrow \\ \mathbb{S}/\eta \otimes \mathbb{F}_2 & \xrightarrow{f\otimes\mathbb{F}_2} & \mathrm{ku} \otimes \mathbb{F}_2. \end{array}$$

The upper horizontal map, and hence the lower horizontal map also, are 3-connected, so to prove the right vertical map kills β, we may equivalently prove that the left vertical map kills the degree 2 generator. To see this, we consider the induced map of LESs

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \pi_2(\mathbb{S}) & \longrightarrow & \pi_2(\mathbb{S}/\eta) & \longrightarrow & \pi_2(\Sigma^2\mathbb{S}) & \xrightarrow{\eta} & \pi_2(\Sigma\mathbb{S}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \pi_2(\mathbb{F}_2) & \longrightarrow & \pi_2(\mathbb{S}/\eta \otimes \mathbb{F}_2) & \longrightarrow & \pi_2(\Sigma^2\mathbb{F}_2) & \longrightarrow & \pi_2(\Sigma\mathbb{F}_2) & \longrightarrow & \dots . \end{array}$$

Identifying these groups, we obtain

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{0} & 2\mathbb{Z} & \hookrightarrow & \mathbb{Z} & \xrightarrow{\eta} & \mathbb{Z}/2 & \xrightarrow{0} & \dots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{mod}\ 2} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \pi_2(\mathbb{S}/\eta \otimes \mathbb{F}_2) & \xrightarrow{\sim} & \mathbb{F}_2 & \longrightarrow & 0 & \longrightarrow & \dots, \end{array}$$

which proves the desired conclusion. Thus, rotating the original exact sequence and arguing as before, we obtain that $\mathrm{ku} \to \mathbb{Z}$ is injective on $\mathbb{F}_2$-homology and identifies with the kernel of $\mathbb{Z} \to \Sigma^3\mathrm{ku} \to \Sigma^3\mathbb{Z} \to \Sigma^3\mathbb{F}_2$. Applying naturality to the diagram of exact sequences

$$\begin{array}{ccccc} \Sigma^2\mathrm{ku} & \xrightarrow{\beta} & \mathrm{ku} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^2\mathbb{Z} & \longrightarrow & \tau_{[0,2]}\mathrm{ku} & \longrightarrow & \mathbb{Z} \end{array}$$

identifies this composite with $\mathbb{Z} \to \Sigma^3\mathbb{Z} \to \Sigma^3\mathbb{F}_2$, where the first map is the first k-invariant of $\tau_{[0,2]}\mathrm{ku} \simeq \tau_{[0,2]}\mathbb{S}/\eta$ (Lemma 3.2). The next Proposition identifies this with the composite $\mathbb{Z} \to \mathbb{F}_2 \to \Sigma^3\mathbb{F}_2$ of reduction $\mod 2$ and the Steenrod square Sq^3. Then, again arguing as before, the kernel of $(\mathrm{Sq}^3)_*$ on $\mathbb{F}_2$-homology is given by contracting against the dual of Sq^3, which is ζ_2 by the above computation and thus kills precisely the monomials in which, additionally, ζ_2 appears to an even power, as desired. □

Proposition 3.4. *The composite* $\mathbb{Z} \to \Sigma^3\mathbb{Z} \to \Sigma^3\mathbb{F}_2$ *of the first* k*-invariant of* $\mathbb{S}/\eta$ *and reduction* $\mod 2$ *agrees with the composite* $\mathbb{Z} \to \mathbb{F}_2 \to \Sigma^3\mathbb{F}_2$ *of reduction* $\mod 2$ *and the Steenrod square* Sq^3.

Proof. The homotopy classes of maps $\mathbb{Z} \to \Sigma^3\mathbb{F}_2$ are classified by $H^3(\mathbb{Z};\mathbb{F}_2)$, which is dual to $H_3(\mathbb{Z};\mathbb{F}_2) \cong \mathbb{Z}\{\zeta_2\}$ and thus 1-dimensional. The second composite we describe is the non-trivial element by the previous calculation(s), so it suffices to show the former composite is non-trivial. To do so, we calculate that it is non-trivial on $\mathbb{F}_2$-homology by using the LES associated to $\tau_{[0,2]}\mathbb{S}/\eta \to \mathbb{Z} \to \Sigma^3\mathbb{Z}$, which yields

$$\pi_3(\mathbb{Z}\otimes\mathbb{F}_2) \xrightarrow{\partial} \pi_3(\Sigma^3\mathbb{Z}\otimes\mathbb{F}_2) \longrightarrow \pi_2(\tau_{[0,2]}\mathbb{S}/\eta\otimes\mathbb{F}_2) \longrightarrow \pi_2(\mathbb{Z}\otimes\mathbb{F}_2) \longrightarrow \pi_2(\Sigma^3\mathbb{Z}\otimes\mathbb{F}_2) = 0.$$

The group $\pi_2(\mathbb{Z}\otimes\mathbb{F}_2) \cong \mathbb{F}_2\{\zeta_1^2\}$ has already been calculated, and a connectivity argument yields

$$\pi_2(\tau_{[0,2]}\mathbb{S}/\eta\otimes\mathbb{F}_2) \cong \pi_2(\mathbb{S}/\eta\otimes\mathbb{F}_2) \cong \pi_1(\Sigma\mathbb{S}\otimes\mathbb{F}_2) \cong \mathbb{F}_2.$$

Thus, the boundary map ∂ cannot be trivial. □

Remark 3.5. The argument already shows that the first k-invariant $\mathbb{Z} \to \Sigma^3\mathbb{Z}$ is non-trivial on $\mathbb{F}_2$-homology. This map is classified by an element of $H^3(\mathbb{Z};\mathbb{Z})$, which turns out to be again $\mathbb{F}_2$. The non-trivial element of this group can also be described as the composite $\mathbb{Z} \to \mathbb{F}_2 \to \Sigma^2\mathbb{F}_2 \to \Sigma^3\mathbb{Z}$ of reduction $\bmod 2$, the Steenrod square Sq^2 and the integral Bockstein β. This element is indeed non-trivial, because its $\bmod 2$ reduction agrees with reduction $\bmod 2$ followed by the non-trivial Steenrod square $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$.

To compute $H^3(\mathbb{Z};\mathbb{Z})$, we first make two general observations on the integral (dual) Steenrod algebra: the homology groups $H_*(\mathbb{Z};\mathbb{Z})$ are a priori finitely generated (i.e. $\mathbb{Z}\otimes\mathbb{Z}$ is finite type) and vanish rationally in positive degrees, since $(\mathbb{Z}\otimes\mathbb{Z})\otimes\mathbb{Q} \simeq \mathbb{Q}$ is Eilenberg-MacLane, hence they are torsion in positive degrees. Thus the UCT gives $H^{n+1}(\mathbb{Z};\mathbb{Z}) \cong \mathrm{Ext}^1(H_n(\mathbb{Z};\mathbb{Z}),\mathbb{Z}) \cong H_n(\mathbb{Z};\mathbb{Z})$ for all $n \geq 1$ (and $H^1(\mathbb{Z};\mathbb{Z}) = 0$). Thus, we are reduced to calculating $H_2(\mathbb{Z};\mathbb{Z})$. Now, the LES associated to $\tau_{\geq 1}\mathbb{S} \to \mathbb{S} \to \mathbb{Z}$ yields that $\pi_2(\mathbb{Z}\otimes\mathbb{Z}) \cong \pi_1(\tau_{\geq 1}\mathbb{S}\otimes\mathbb{Z})$, but the LES associated to $\tau_{\geq 2}\mathbb{S} \to \tau_{\geq 1}\mathbb{S} \to \Sigma\mathbb{F}_2$ yields that $\pi_1(\tau_{\geq 1}\mathbb{S}\otimes\mathbb{Z}) \cong \pi_1(\Sigma\mathbb{F}_2\otimes\mathbb{Z}) \cong \mathbb{F}_2$, as desired.

Proposition 3.6. *The map* $c\colon \mathrm{ko} \to \mathrm{ku}$ *induces on* $\mathbb{F}_2$*-homology the inclusion* $\mathbb{F}_2[\zeta_1^4,\zeta_2^2,\zeta_3,\dots] \hookrightarrow \mathbb{F}_2[\zeta_1^2,\zeta_2^2,\zeta_3,\dots]$.

Proof. To prove this, we consider the Wood cofiber sequence $\Sigma\mathrm{ko} \to \mathrm{ko} \to \mathrm{ku}$ of Corollary 2.6, where the first map is multiplication by the Hopf element $\eta \in \pi_1(\mathbb{S})$. This vanishes upon tensoring with $\mathbb{F}_2$, because any $\mathbb{F}_2$-module is a generalized Eilenberg-MacLane spectrum, so again arguing as before, we obtain that $\mathrm{ko} \to \mathrm{ku}$ is injective on $\mathbb{F}_2$-homology and identifies with the kernel of $\mathrm{ku} \to \Sigma^2\mathrm{ko} \to \Sigma^2\mathrm{ku} \to \Sigma^2\mathbb{F}_2$. Applying naturality to the diagram of exact sequences

$$\begin{array}{ccccc} \Sigma\mathrm{ko} & \xrightarrow{\eta} & \mathrm{ko} & \xrightarrow{c} & \mathrm{ku} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma\mathbb{F}_2 & \longrightarrow & \tau_{[0,1]}\mathrm{ko} & \longrightarrow & \mathbb{Z} \end{array}$$

identifies this composite with $\mathrm{ku} \to \mathbb{Z} \to \Sigma^2\mathbb{F}_2$, where the latter map is the zeroth k-invariant of $\tau_{[0,1]}\mathrm{ko} \simeq \tau_{[0,1]}\mathbb{S}$. The next Proposition identifies this with the composite $\mathbb{Z} \to \mathbb{F}_2 \to \Sigma^2\mathbb{F}_2$ of reduction $\bmod 2$ and the Steenrod square Sq^2. Then, arguing one last time as before, the kernel of $(\mathrm{Sq}^2)_*$ on $\mathbb{F}_2$-homology is given by contracting against the dual of Sq^2, which is ζ_1^2 by the above computation and thus kills precisely the monomials in which additionally ζ_1 appears to a power divisible by 4, as desired. □

Proposition 3.7. *The zeroth* k*-invariant* $\mathbb{Z} \to \Sigma^2\mathbb{F}_2$ *of the sphere spectrum* $\mathbb{S}$ *agrees with the composite* $\mathbb{Z} \to \mathbb{F}_2 \to \Sigma^2\mathbb{F}_2$ *of reduction* $\bmod 2$ *and the Steenrod square* Sq^2.

Proof. The homotopy classes of maps $\mathbb{Z} \to \Sigma^2\mathbb{F}_2$ are classified by $H^2(\mathbb{Z};\mathbb{F}_2)$, which is dual to $H_2(\mathbb{Z};\mathbb{F}_2) \cong \mathbb{Z}\{\zeta_1^2\}$ and thus 1-dimensional. The second composite we describe is the non-trivial element by the previous calculation(s), so it suffices to show the former composite is non-trivial. To do so, we calculate that it is non-trivial on $\mathbb{F}_2$-homology by using the LES associated to $\tau_{[0,1]}\mathbb{S} \to \mathbb{Z} \to \Sigma^2\mathbb{F}_2$, which yields

$$\pi_2(\tau_{[0,1]}\mathbb{S}\otimes\mathbb{F}_2) \longrightarrow \pi_2(\mathbb{Z}\otimes\mathbb{F}_2) \xrightarrow{\partial} \pi_2(\Sigma^2\mathbb{F}_2\otimes\mathbb{F}_2) \longrightarrow \pi_1(\tau_{[0,1]}\mathbb{S}\otimes\mathbb{F}_2).$$

That $\pi_{>0}(\mathbb{S}\otimes\mathbb{F}_2)=0$ implies $\pi_i(\tau_{[0,1]}\mathbb{S}\otimes\mathbb{F}_2)\cong\pi_{i-1}(\tau_{\geq 2}\mathbb{S}\otimes\mathbb{F}_2)$ for $i\geq 1$. In particular, this group vanishes for $i=1,2$ by connectivity reasons. However, the group $\pi_2(\Sigma^2\mathbb{F}_2\otimes\mathbb{F}_2)\cong\mathbb{F}_2$ is non-trivial, so the boundary map ∂ cannot be trivial. □

Remark 3.8. For the non-connective K-theory spectra, it follows from chromatic homotopy theory that $\mathrm{KU}\otimes\mathbb{Z}$ is a rational spectrum, hence $\mathrm{KU}\otimes\mathbb{F}_2\simeq(\mathrm{KU}\otimes\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2\simeq 0$. The Wood cofiber sequence implies that $\mathrm{KO}\otimes\mathbb{F}_2$ is a retract of $\mathrm{KU}\otimes\mathbb{F}_2\simeq 0$ and thus vanishes as well. In total, the $\mathbb{F}_2$-homology of both KU and KO is trivial.

4. Tate L-theory of ku and ko

In this section, we compute the Tate L-theory spectra of ku and ko. First, we make some recollections.

Let R be a connective $\mathbb{E}_\infty$-ring spectrum. The homotopy groups of the spectra R^{hC_2} and R^{tC_2} (w.r.t. the trivial C_2-action) are computed by a homotopy fixed point resp. Tate spectral sequence

$$E_1^{p,q}=H^{-q}(C_2;\pi_p(R))\Rightarrow\pi_{p+q}(R^{hC_2})\quad\text{resp.}\quad\hat{E}_1^{p,q}=\hat{H}^{-q}(C_2;\pi_p(R))\Rightarrow\pi_{p+q}(R^{tC_2}).$$

(This is the Serre grading with cohomological indexing on the filtration degree and homological indexing on the total degree, i.e. the r-th differential has type $(+r,-r-1)$.) These are conditionally convergent (right) half-plane spectral sequences with entering differentials and there is a homomorphism of spectral sequences $E\to\hat{E}$ converging to the action of the canonical map $\mathrm{can}\colon R^{hC_2}\to R^{tC_2}$ on homotopy groups. The comparison result of Boardman then implies that $\pi_*(\mathrm{can})\colon\pi_*(R^{hC_2})\xrightarrow{\sim}\pi_*(R^{tC_2})$ is an isomorphism for $*<0$ (indeed, the canonical map is an isomorphism on the E_2-page in complementary degree $q<0$ and a surjection in degree $q=0$). The composition $R^{hC_2}\to R\to R^{tC_2}$ of the projection proj and the Tate-valued Frobenius φ_2 on the other hand induces the zero map on negative homotopy groups since it factors through the connective spectrum R.

Now, recall that there exists a canonical *Tate Poincaré structure* on $\mathrm{Perf}(R)$ (cf. [Cal+22, Example 3.2.12]), whose associated L-theory spectrum is *Tate L-theory* $L^{\mathrm{t}}(R)$. There is, as always, a tautological cofiber sequence $L^{\mathrm{q}}(R)\to L^{\mathrm{t}}(R)\to L^{\mathrm{tn}}(R)$ and the latter term, *Tate normal L-theory*, turns out to be explicitly computable by the equalizer formula of Harpaz–Nikolaus–Shah [HNS]. Together with the preceding considerations, this gives

$$L^{\mathrm{tn}}(R)\simeq\mathrm{Eq}\left[R^{hC_2}\underset{\mathrm{can}}{\overset{\varphi_2\mathrm{proj}}{\rightrightarrows}}R^{tC_2}\right]\simeq\mathrm{Eq}\left[\tau_{\geq 0}R^{hC_2}\underset{\mathrm{can}}{\overset{\varphi_2\mathrm{proj}}{\rightrightarrows}}\tau_{\geq 0}R^{tC_2}\right].$$

In particular, $L^{\mathrm{tn}}_*(R)=0$ for $*<-1$.

The complex case. The connective complex K-theory spectrum ku is even and has no 2-torsion in its homotopy groups, so the E_1-page of the homotopy fixed point resp. Tate spectral sequence is concentrated in even coordinates, so they immediately collapse and are thus a fortiori strongly convergent by Boardman's result. In particular, ku^{hC_2} resp. ku^{tC_2} are again even $\mathbb{E}_\infty$-ring spectra.

In a more precise fashion, there is a Bott element $\beta\in E_1^{2,0}=H^0(C_2;\pi_2(\mathrm{ku}))=\pi_2(\mathrm{ku})=\mathbb{Z}$ and a Tate element $t\in E_1^{0,-2}=H^2(C_2;\pi_0(\mathrm{ku}))=\mathbb{Z}/2$, which together generate the $E_1=E_\infty$-page as a bigraded polynomial $\mathbb{Z}$-algebra with the relation $2t=0$. For now, we will solve one particular extension problem on this page and then leverage it, namely the one associated to the SES

$$0\longrightarrow E_\infty^{2,-4}\longrightarrow\pi_{-2}(\mathrm{ku}^{hC_2})/F^4\pi_{-2}(\mathrm{ku}^{hC_2})\longrightarrow E_\infty^{0,-2}\longrightarrow 0.$$

The extension $\mathbb{Z}\xrightarrow{2\cdot}\mathbb{Z}\to\mathbb{Z}/2$ induces a boundary map $\delta\colon B\mathbb{Z}/2\to B^2\mathbb{Z}\simeq BU(1)$ classifying a complex line bundle over $B\mathbb{Z}/2$ with vanishing tensor square. Its Chern class $c:=c_1(\delta)\in H^2(BC_2;\mathrm{ku})=\pi_{-2}(\mathrm{ku}^{hC_2})$ w.r.t. the appropriate complex orientation of ku then satisfies the identity (by virtue of the formal group law of complex K-theory)

$$0=c_1(\delta^{\otimes 2})=2c_1(\delta)-\beta c_1(\delta)^2.$$

The projection onto the $E_\infty^{0,-2}$-term corresponds to the change-of-coefficients $\mathrm{ku}^{hC_2}\to\mathbb{Z}^{hC_2}$, so that the image of c is (up to a sign) the first Chern class in integral singular cohomology of the bundle classified by δ, which is tautologically

non-trivial. Thus, the image of c in $\pi_{-2}(\mathrm{ku}^{hC_2})/F^4\pi_{-2}(\mathrm{ku}^{hC_2})$ lifts the generator of $E_\infty^{0,-2}$ on one hand and, on the other hand, its product with 2 equals βc^2, which is contained in the image of $E_\infty^{2,-4}$ as a generator by multiplicativity of the spectral sequence (since the Bott element lives in filtration degree 2 and reduces to β on the E_∞-page, and c reduces to the Tate element t on the E_∞-page as just observed). This implies the the extension is non-trivial, so we obtain $\pi_{-2}(\mathrm{ku}^{hC_2})/F^4\pi_{-2}(\mathrm{ku}^{hC_2}) = \mathbb{Z}/4$ by the classification of extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/2$.

Now, let's shift attention to the Tate spectral sequence $\hat{E}$, whose $\hat{E}_1 = \hat{E}_\infty$-page is generated as a bigraded $\mathbb{Z}/2$-algebra by the image of β in $\hat{E}_1^{2,0} = \hat{H}^0(C_2;\pi_2(\mathrm{ku})) = \mathbb{Z}/2$ and the Tate elements $t^{\pm 1} \in \hat{E}_1^{0,\mp 2} = \mathbb{Z}/2$ with the obvious relation $tt^{-1} = 1$. The multiplication by the image $c \in \pi_{-2}(\mathrm{ku}^{tC_2})$ of the Chern class thus induces isomorphisms on $\pi_*(\mathrm{ku}^{tC_2})$, because it is compatible with the complete, exhaustive filtrations and induces the invertible multiplication with the Tate element on the graded pieces. In particular, there is an inverse element $c^{-1} \in \pi_2(\mathrm{ku}^{tC_2})$. The multiplicative structure on the $\hat{E}_1 = \hat{E}_\infty$-page then immediately implies that $\pi_*(\mathrm{ku}^{tC_2}) = \pi_0(\mathrm{ku}^{tC_2})[c^{\pm 1}]$ as graded rings, so it remains to identify the abelian group $\pi_0(\mathrm{ku}^{tC_2})$.

After multiplying the previous computation by c^{-1} in $\hat{E}$, it provides a solution to the first extension problem in the induced filtration of $\pi_0(\mathrm{ku}^{tC_2})$. This argument shows that the element 2 has filtration degree exactly 2. The filtration is complete, so we obtain an induced map $\mathbb{Z}_2 \to \pi_0(\mathrm{ku}^{tC_2})$, $1 \mapsto 1$. This is a map of completely, exhaustively filtered abelian groups and induces an isomorphism $\mathbb{Z}/2 \xrightarrow{\sim} E_\infty^{2n,-2n}$, $[1] \mapsto (\beta t)^n$ on the graded pieces by the multiplicative structure of the $\hat{E}_\infty$-page, hence is an isomorphism $\mathbb{Z}_2 \xrightarrow{\sim} \pi_0(\mathrm{ku}^{tC_2})$.

Let us now return to the homotopy fixed point spectral sequence. There is the canonical map of spectral sequences $\mathrm{can}\colon E \to \hat{E}$ and there is another map of spectral sequences $E \to \hat{E}$ converging to the action of $\varphi_2\mathrm{proj}$ on homotopy groups. The latter is obtained from the Postnikov-filtrations $(\tau_{\geq\bullet}R)^{hC_2} \to \tau_{\geq\bullet}R \to (\tau_{\geq\bullet}R)^{tC_2}$, where the latter transformation is given by the Tate-valued Frobenius (which is defined for all non-unital rings). Note that this transformation induces 0 on the E_1-page in complementary degree $q < 0$ (for connectivity reasons) and also in complementary degree $q = 0$ and filtration degree $p > 0$ (since the multiplication in a ring spectrum adds degrees). Then we can import these extension problems back along the map $\mathrm{can} - \varphi_2\mathrm{proj}\colon E \to \hat{E}$ to identify $\pi_*(\mathrm{ku}^{hC_2})$ as well as the induced map $\pi_*(\mathrm{can} - \varphi_2\mathrm{proj})$. In even degrees $2n < 0$, this is an isomorphism (as was already clear on general grounds). In the even degree 0, we obtain the diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^2\pi_0(\mathrm{ku}^{hC_2}) & \longrightarrow & \pi_0(\mathrm{ku}^{hC_2}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \wr} & & \downarrow & & \downarrow{\scriptstyle 1-F} & & \\
0 & \longrightarrow & F^2\pi_0(\mathrm{ku}^{tC_2}) & \longrightarrow & \pi_0(\mathrm{ku}^{tC_2}) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}$$

The bottom filtration has already been identified, so the composite $\mathbb{Z} \to \pi_0(\mathrm{ku}^{hC_2}) \to \pi_0(\mathrm{ku}^{tC_2}) \cong \mathbb{Z}_2$ factors as a map $\alpha\colon \mathbb{Z} \to 2\mathbb{Z}_2$ (since $1 - F = 0$) s.t. $\pi_0(\mathrm{can} - \varphi_2\mathrm{proj})$ identifies with the map $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$, $(k, x) \mapsto \alpha(k) + 2x$. The composition of this with the shear automorphism $(k, x) \mapsto (k, x - \alpha(k)/2)$ yields the canonical form $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$, $(k, x) \mapsto 2x$. In even degrees $2n > 0$, we instead obtain the diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{2n+2}\pi_{2n}(\mathrm{ku}^{hC_2}) & \longrightarrow & \pi_{2n}(\mathrm{ku}^{hC_2}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \wr} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & F^{2n+2}\pi_{2n}(\mathrm{ku}^{tC_2}) & \longrightarrow & F^{2n}\pi_{2n}(\mathrm{ku}^{tC_2}) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}$$

In this case, the composite $\alpha\colon \mathbb{Z} \to \pi_{2n}(\mathrm{ku}^{hC_2}) \to F^{2n}\pi_{2n}(\mathrm{ku}^{tC_2}) \cong 2^n\mathbb{Z}_2$ is given by $k \mapsto 2^n\alpha(1)i(k)$, where $i\colon \mathbb{Z} \to \mathbb{Z}_2$ is the canonical inclusion and $\alpha(1) \in \mathbb{Z}_2^\times$, because it has a non-zero reduction mod 2. Then $\pi_{2n}(\mathrm{can} - \varphi_2\mathrm{proj})$ identifies with the map $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$, $(k, x) \mapsto 2^n(\alpha(1)i(k) + 2x)$. Twisting this by the unit $\alpha(1)$ in source and target yields the canonical form $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$, $(k, x) \mapsto 2^n(i(k) + 2x)$.

The Tate normal L-theory spectrum $L^{\mathrm{tn}}(\mathrm{ku})$ identifies with the fiber of $\mathrm{can}-\varphi_2\mathrm{proj}\colon \mathrm{ku}^{hC_2} \to \mathrm{ku}^{tC_2}$ by the equalizer formula. These spectra are even, so the induced LES reduces to computing (co-)kernels and we immediately obtain

$$L^{\mathrm{tn}}_k(\mathrm{ku}) = \begin{cases} \mathbb{Z}, & k = 2n,\ n \geq 0, \\ \mathbb{Z}/2^{n+1}, & k = 2n+1,\ n \geq 0, \\ \mathbb{Z}/2, & k = -1, \\ 0, & k \leq -2. \end{cases}$$

The algebraic π-π Theorem implies that $L^{\mathrm{q}}(\mathrm{ku}) \xrightarrow{\sim} L^{\mathrm{q}}(\mathbb{Z})$ and the homotopy groups of the latter are well-known. Then running the LES induced by the fiber sequence $L^{\mathrm{q}}(\mathrm{ku}) \to L^{\mathrm{t}}(\mathrm{ku}) \to L^{\mathrm{tn}}(\mathrm{ku})$ yields

$$L^{\mathrm{t}}_k(\mathrm{ku}) = \begin{cases} \mathbb{Z} \oplus 8\mathbb{Z}, & k = 4n,\ n \geq 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2, & k = 4n+2,\ n \geq 0, \\ \mathbb{Z}/2^{n+1}, & k = 2n+1,\ n \geq 0, \\ 8\mathbb{Z}, & k = 4n,\ n < 0, \\ \mathbb{Z}/2, & k = 4n+2,\ n < -1, \\ 0, & \text{else}. \end{cases}$$

(The only non-trivial part of this calculation is to identify the differentials $\partial\colon L^{\mathrm{tn}}_{2n+1}(\mathrm{ku}) \to L^{\mathrm{q}}_{2n}(\mathrm{ku})$ for $n \geq 0$ resp. the differential $\partial\colon L^{\mathrm{tn}}_{-1}(\mathrm{ku}) \to L^{\mathrm{q}}_{-2}(\mathrm{ku})$, which vanish resp. is an isomorphism by using naturality for the $\mathbb{E}_\infty$-ring map $\mathrm{ku} \to \mathbb{Z}$ and quoting [LNP, Lemma 2.8] and [LNP, Remark 2.7] respectively.)

The real case. For the connective real K-theory spectrum ko, it seems quite hard to do the computations directly using the spectral sequence. Instead, we start with the following observation: there is a composite of $\mathbb{E}_\infty$-ring maps

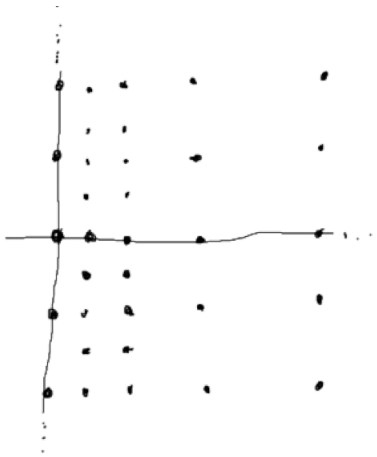
$$\mathrm{MSO} \to L^s(\mathbb{Z}) \to K(\mathbb{Z})^{tC_2} \to K(\mathbb{R})^{tC_2} \to \mathrm{ko}^{tC_2},$$

in which the maps are the Ranicki-Sullivan orientation, the map appearing in the fundamental fiber square of Hermitian K-theory, the algebraic K-theory change-of-rings along $\mathbb{Z} \to \mathbb{R}$ and, finally, the canonical comparison between algebraic and topological K-theory. Note that the C_2-action on $K(\mathbb{Z})$ (resp. $K(\mathbb{R})$) is induced by the Hermitian duality on $\mathrm{Perf}(\mathbb{Z})$ (resp. $\mathrm{Perf}(\mathbb{R})$) and turns out to be non-trivial (cf. [Cal+26, Lemma 3.2.4]). The corresponding C_2-action on ko, however, is trivial. We offer two reasons (better safe than sorry):

- The C_2-action on ko is induced by the Hermitian duality $\mathrm{Vect}_{\mathbb{R}} \simeq \mathrm{Vect}^{\mathrm{op}}_{\mathbb{R}}$ given by taking duals (i.e. transposing matrices). This induces a C_2-action on groupoid cores via the canonical identification $(\mathrm{Vect}^{\mathrm{op}}_{\mathbb{R}})^{\simeq} \simeq \mathrm{Vect}^{\simeq}_{\mathbb{R}}$ induced by taking inverses. Furthermore, this groupoid is $\bigsqcup_{n\geq 0} BO(n)$ and $A = A^{-t}$ for all orthogonal matrices $A \in O(n)$, so the induced C_2-action on $\mathrm{Vect}^{\simeq}_{\mathbb{R}}$ and hence on its group completion ko is trivial.
- It turns out that, generally, every C_2-action on ko is trivial, see Appendix A.

Thus ko^{tC_2} is an MSO-module, but it is also a priori 2-complete (it is the Tate construction on a bounded below spectrum), so it is in particular an $\mathrm{MSO}_{(2)}$-module. Then, the Mahowald $\mathbb{E}_2$-ring map $\mathbb{Z}_{(2)} \to \mathrm{MSO}_{(2)}$ implies that ko^{tC_2} is a $\mathbb{Z}$-module, and hence a generalized Eilenberg-MacLane spectrum. In particular, the Hopf element η acts trivially on ko^{tC_2} and thus the Wood cofiber sequence splits after applying the exact functor $(-)^{tC_2}$, yielding an equivalence $\mathrm{ku}^{tC_2} \simeq \mathrm{ko}^{tC_2} \oplus \Sigma^2\mathrm{ko}^{tC_2}$, where the first summand inclusion is induced by the complexification map $c\colon \mathrm{ko} \to \mathrm{ku}$. Then using that $\mathbb{Z}_2$ is indecomposable as an abelian group and the multiplicative structure, we can reverse-engineer that $\pi_*(\mathrm{ko}^{tC_2}) = \mathbb{Z}_2[x^{\pm 1}]$ on a generator in degree $|x| = -4$.

Now, let's reverse-engineer the entire Tate spectral sequence of ko. The $\hat{E}_1$-page is periodic in both directions and looks as follows (each black dot represents a $\mathbb{Z}/2$):



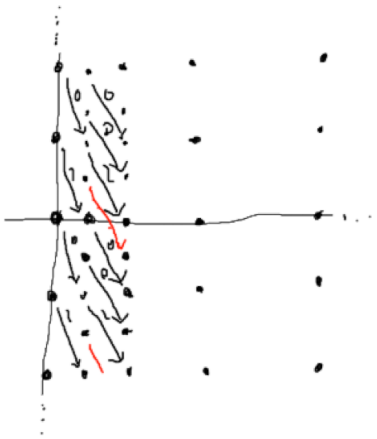
First, note that all terms on the $\hat{E}_1$-page are finite, hence the derived $\hat{E}_\infty$-page automatically vanishes and the Tate spectral sequence is (strongly) convergent by Boardman's result. Now, consider the realification map $r\colon \mathrm{ku} \to \mathrm{ko}$, which induces a map of Tate spectral sequences that hits all the elements in degrees $(8n+2, 2k)$ and $(8n+4, 2k)$ with $n \geq 0$, $k \in \mathbb{Z}$. The Tate spectral sequence for ku^{tC_2} collapses immediately, hence all differentials leaving these entries vanish. Furthermore, there is the canonical map $\mathrm{can}\colon E \to \hat{E}$ and the former has no differentials leaving the $q = 0$ row, because ko canonically splits off of ko^{hC_2}. Thus, all differentials leaving the 0-th row in $\hat{E}$ vanish. In particular, $d_1(\beta_{\mathbb{R}}) = 0$ (here, $\beta_{\mathbb{R}} \in \hat{E}_1^{8,0}$ is the canonical image of the real Bott element) and thus multiplication by $\beta_{\mathbb{R}}$ makes the $\hat{E}_1$-page horizontally 8-periodic to the right.

For any $k \geq 0$, the target $\pi_*(\mathrm{ko}^{tC_2})$ is trivial in degree $4k+2$, so the term $\hat{E}_1^{2,4k}$ cannot survive to the $\hat{E}_\infty$-page, but since every differential leaving this position vanishes and $\hat{E}_1^{0,4k+3} = 0$, this forces $d_1\colon \hat{E}_1^{1,4k+2} \xrightarrow{\sim} \hat{E}_1^{2,4k}$ to be an isomorphism. The identity $d_1^2 = 0$ then forces $d_1\colon \hat{E}_1^{0,4k+4} \to \hat{E}_1^{1,4k+2}$ to be the zero map. In particular, multiplication by the generators in bidegrees $(0, \pm 4)$ exhibits the $\hat{E}_1$-page as vertically 4-periodic in both directions.

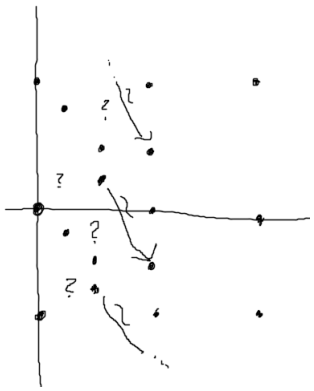
Next, let's consider the term $\hat{E}_1^{1,0}$, which cannot survive to the E_∞-page either for the same reason. The differentials leaving this position all vanish, so this forces $d_1\colon \hat{E}_1^{0,2} \xrightarrow{\sim} \hat{E}_1^{1,0}$ to be an isomorphism. The identity $d_1^2 = 0$ then forces $d_1\colon \hat{E}_1^{1,0} \to \hat{E}_1^{2,-2}$ to be the zero map. These differentials prolong vertically by the 4-periodicity.

Finally, let's consider the term $\hat{E}_1^{4,-2}$, which also cannot survive to the $\hat{E}_\infty$-page for these reasons. The differentials leaving this position all vanish—the 3-rd column is trivial, the term $\hat{E}_2^{1,2} = 0$ by the previous argument and $\hat{E}_1^{0,3} = 0$ already—so this forces $d_2\colon \hat{E}_2^{2,1} \xrightarrow{\sim} \hat{E}_2^{4,-2}$ to be an isomorphism as the only remaining way to kill this term. In particular, $\hat{E}_1^{2,1}$ has to survive to the next page, so the differential $d_1\colon \hat{E}_1^{1,3} \to \hat{E}_1^{2,1}$ is zero. This differential prolongs vertically by the 4-periodicity.

In total, we have identified all the potentially interesting d_1-differentials with one exception (marked in red):



These considerations lead to the following picture of the E_2-page:



In particular, the identity $d_2^2 = 0$ forces the differentials $d_2 \colon \hat{E}_2^{0,4k} \to \hat{E}_2^{2,4k-3}$ to vanish for all $k \in \mathbb{Z}$. Together with the previous observations, this implies that the $\hat{E}_2$-page again enjoys the same periodicity properties as the $\hat{E}_1$-page did. The ?-terms sit in total degree $2 \mod 4$ in the $1 \mod 8$-columns and in total degree $1 \mod 4$ in the $2 \mod 8$-columns, and all other terms sit in total degree $0 \mod 4$. Thus, the only potentially non-trivial differentials that remain are those leaving the ?-terms.

In particular, the terms in the 0-th column survive to the $\hat{E}_\infty$-page and thus *every* page enjoys the vertical periodicity. The horizontal periodicity is enjoyed, by the same logic, by the pages until the $\hat{E}_6$-page, which has a potentially non-trivial differential $d_6 \colon \hat{E}_6^{2,7} \to \hat{E}_6^{8,0}$ that could kill $\beta_{\mathbb{R}}$. However, if this were the case, periodicity on the $\hat{E}_6$-page would imply that all ?-terms in the $2 \mod 8$-columns get killed on this page. Then, for degree reasons, the ?-terms in the $1 \mod 8$-columns would survive to the $\hat{E}_\infty$-page, contradicting that $\pi_*(\mathrm{ko}^{tC_2})$ is trivial in these degrees. Thus, all subsequent pages enjoy the horizontal periodicity as well. However, the same logic then applies to any differential leaving the ?-terms in the 2-nd column, so we deduce those differentials are all trivial. It follows that the ?-terms in the 2-nd column could only have been killed all the way back on the $\hat{E}_1$-page by the differentials $d_1 \colon \hat{E}_1^{1,4k+1} \to \hat{E}^{2,4k-1}$, which thus have to be isomorphisms for all $k \in \mathbb{Z}$, and this prolongs horizontally by the 8-periodicity. This finishes the identification of all the differentials. In particular, the spectral sequence collapses on the $\hat{E}_3$-page.

Having completely understood the Tate spectral sequence, we can import this knowledge back to the homotopy fixed point spectral sequence for ko^{hC_2}. The E_1-page of this consists of the fourth quadrant of the Tate spectral sequence, yet with a larger $q = 0$ row, which however splits off as a copy of $\pi_*(\mathrm{ko})$. The resulting difference in the E_∞-page is that the $(8n+1, 0)-$, $(8n+2, 0)-$, $(8n+2, -1)-$ and $(8n+4, -2)$-terms for all $n \geq 0$ survive this time. In particular, we obtain

$$\pi_k(\mathrm{ko}^{hC_2}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & k = 8n,\ 8n+4,\ n \geq 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & k = 8n+1,\ 8n+2,\ n \geq 0, \\ \mathbb{Z}_2, & k = 4n,\ n < 0, \\ 0, & \text{else.} \end{cases}$$

Then, analogously to the ku-case considered previously, we can use this to also identify the map induced by $\mathrm{can} - \varphi_2 \mathrm{proj}$ on homotopy groups in degrees divisible by 4. In degrees $4n < 0$, this is an isomorphism (as was already clear on general grounds). In the degree $n = 0$, it is again isomorphic to the map $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2,\ (k, x) \mapsto 2x$. Finally, in degrees $8n > 0$ resp. $8n + 4 > 0$ it is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2,\ (k, x) \mapsto 2^{4n}(i(k) + 2x)$ resp. $2^{4n+3}(i(k) + 2x)$. The LES induced

by can $-\varphi_2$proj reduces to isomorphisms and (co-)kernel computations for degree reasons and we immediately obtain

$$L^{\mathrm{tn}}_k(\mathrm{ko}) = \begin{cases} \mathbb{Z}, & k = 4n,\ n \geq 0, \\ (\mathbb{Z}/2)^2, & k = 8n+1, 8n+2,\ n \geq 0, \\ \mathbb{Z}/2^{4n+3}, & k = 8n+3,\ n \geq 0, \\ \mathbb{Z}/2^{4n+4}, & k = 8n+7,\ n \geq 0, \\ \mathbb{Z}/2, & k = -1, \\ 0, & \text{else.} \end{cases}$$

Then, we may use the same strategy as in the complex case to compute Tate L-theory, yielding

$$L^{\mathrm{t}}_k(\mathrm{ko}) = \begin{cases} \mathbb{Z} \oplus 8\mathbb{Z}, & k = 4n,\ n \geq 0 \\ (\mathbb{Z}/2)^2, & k = 8n+1,\ n \geq 0 \\ \mathbb{Z}/2 \oplus (\mathbb{Z}/2)^2, & k = 8n+2,\ n \geq 0, \\ \mathbb{Z}/2^{4n+3}, & k = 8n+3,\ n \geq 0 \\ \mathbb{Z}/2, & k = 8n+6,\ n \geq 0 \text{ or } k = 4n+2,\ n < -1, \\ \mathbb{Z}/2^{4n+4}, & k = 8n+7,\ n \geq 0, \\ 8\mathbb{Z}, & k = 4n,\ n < 0, \\ 0, & \text{else.} \end{cases}$$

(The only additionally non-trivial part of this calculation is to identify the extension $0 \to L^{\mathrm{q}}_{8n+2}(\mathrm{ko}) \to L^{\mathrm{t}}_{8n+2}(\mathrm{ko}) \to L^{\mathrm{tn}}_{8n+2}(\mathrm{ko}) \to 0$ that appears, but this extension admits a splitting that is obtained by using naturality for the $\mathbb{E}_\infty$-ring map $c\colon \mathrm{ko} \to \mathrm{ku}$.)

Remark 4.1. The natural question of computing $L^{\mathrm{t}}(\mathrm{KU})$ resp. $L^{\mathrm{t}}(\mathrm{KO})$ seems much more difficult, because even the quadratic L-groups $L^{\mathrm{q}}(\mathrm{KU})$ resp. $L^{\mathrm{q}}(\mathrm{KO})$ are not clearly accessible.

Appendix A. $\mathbb{E}_\infty$-endomorphisms

In this appendix, we compute the $\mathbb{E}_\infty$-endomorphisms of ko and KO (a similar, but easier, analysis applies to ku and KO as well). First, we analyze the non-connective version KO. We make the following observations/recollections:

- The rationalization $\mathrm{KO}_{\mathbb{Q}}$ is the free rational $\mathbb{E}_\infty$-algebra on a single invertible generator in degree $|\alpha| = 4$. Indeed, the image of α in $\mathrm{KO}_{\mathbb{Q}}$ has inverse given by $\alpha\beta^{-1}_{\mathbb{R}}/4$, so there is an $\mathbb{E}_\infty$-map $\mathrm{Free}_{\mathbb{E}_\infty}(\alpha^{\pm 1}) \to \mathrm{KO}_{\mathbb{Q}}$, which one checks to be an isomorphism on homotopy groups (either side is a Laurent series ring $\mathbb{Q}[\alpha^{\pm 1}]$ on a degree 4 generator).
- For every prime p, the p-completion $\mathrm{KU}^\wedge_p$ is in fact $K(1)$-local at p and $L_{K(1)}\mathbb{S} \to \mathrm{KU}^\wedge_p$ is a $\mathbb{Z}^\times_p$-Galois extension by work of Rognes. Furthermore, $\mathrm{KO} \to \mathrm{KU}$ is a C_2-Galois extension, hence $\mathrm{KO}^\wedge_p \to \mathrm{KU}^\wedge_p$ is a C_2-Galois extension and, in particular, $\mathrm{KO}^\wedge_p \simeq (\mathrm{KU}^\wedge_p)^{hC_2}$ is also $K(1)$-local. Then the Galois correspondence implies that $L_{K(1)}\mathbb{S} \to \mathrm{KO}^\wedge_p$ is a $\mathbb{Z}^\times_p/\{\pm 1\}$-Galois extension.

To compute $\mathrm{End}_{\mathbb{E}_\infty}(\mathrm{KO})$, we take the fracture square of KO and apply $\mathrm{Map}_{\mathrm{CAlg(Sp)}}(\mathrm{KO}, -)$ to obtain the cartesian square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg(Sp)}}(\mathrm{KO},\mathrm{KO}) & \longrightarrow & \prod_{p \text{ prime}} \mathrm{Map}_{\mathrm{CAlg(Sp)}}(\mathrm{KO},\mathrm{KO}^\wedge_p) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg(Sp)}}(\mathrm{KO},\mathrm{KO}_{\mathbb{Q}}) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg(Sp)}}(\mathrm{KO},(\prod_{p \text{ prime}} \mathrm{KO}^\wedge_p)_{\mathbb{Q}}). \end{array}$$

Then using universal properties and the observation $\mathrm{KO}_{\mathbb{Q}} \simeq \mathrm{Free}_{\mathbb{E}_\infty}(\alpha^{\pm 1})$, this becomes the cartesian square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathrm{KO},\mathrm{KO}) & \longrightarrow & \prod_{p \text{ prime}} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}^\wedge_p)}(\mathrm{KO}^\wedge_p,\mathrm{KO}^\wedge_p) \\ \downarrow & \lrcorner & \downarrow \\ (\Omega^{\infty+4}\mathrm{KO}_{\mathbb{Q}})^\times & \longrightarrow & (\Omega^{\infty+4}(\prod_{p \text{ prime}} \mathrm{KO}^\wedge_p)_{\mathbb{Q}})^\times. \end{array}$$

The terms in the lower horizontal row are rational anima whose homotopy groups are concentrated in degrees divisible by 4, and the upper-right term can be calculated using Galois theory as

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}^\wedge_p)}(\mathrm{KO}^\wedge_p,\mathrm{KO}^\wedge_p) \simeq \mathrm{Map}_{\mathrm{CAlg}(L_{K(1)}\mathrm{Sp})}(\mathrm{KO}^\wedge_p,\mathrm{KO}^\wedge_p) \simeq \mathbb{Z}_p^\times/\{\pm 1\}.$$

In particular, this anima is discrete and every $\mathbb{E}_\infty$-ring endomorphism of $\mathrm{KO}^\wedge_p$ is automatically an automorphism. The LES associated to this cartesian square then splits into a bunch of short exact sequences in positive degrees and a pullback square on π_0. The latter is computed as follows:

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \prod_{p \text{ prime}} \mathbb{Z}_p^\times/\{\pm 1\} & \xrightarrow{(-)^2} & \prod_{p \text{ prime}} \mathbb{Z}_p^\times & \longrightarrow & \prod_{p \text{ prime}} \mathbb{Z}_p & \longrightarrow & (\prod_{p \text{ prime}} \mathbb{Z}_p)_{\mathbb{Q}}. \end{array}$$

Here, the bottom composite identifies with the right vertical map in the diagram because the k-th Adams operation for $k \in \mathbb{Z}_p^\times$ acts on the degree 4 generator of $\mathrm{KO}^\wedge_p$ by $\alpha \mapsto k^2\alpha$. Thus, $\mathrm{End}_{\mathbb{E}_\infty}(\mathrm{KO})$ is a connected, rational anima whose only non-trivial homotopy groups are a copy of the quotient $\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}$ of the finite adeles, which is a rational vector space of uncountable dimension, in every positive degree $4n+3$. In particular, every $\mathbb{E}_\infty$-ring endomorphism of KO is automatically an automorphism.

To study the connective cover ko instead, we observe that $\mathrm{End}_{\mathbb{E}_\infty}(\mathrm{ko}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathrm{ko},\mathrm{KO})$ by the universal property of $\tau_{\geq 0}$ and the identity $\mathrm{KO} \simeq \mathrm{ko}[\beta_{\mathbb{R}}^{-1}]$ implies that $\mathrm{End}_{\mathbb{E}_\infty}(\mathrm{KO}) \hookrightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathrm{ko},\mathrm{KO}) \simeq \mathrm{End}_{\mathbb{E}_\infty}(\mathrm{ko})$ is a subanima. We now argue that it is in fact an equivalence, i.e. every $\mathbb{E}_\infty$-ring map $\mathrm{ko} \to \mathrm{KO}$ inverts $\beta_{\mathbb{R}}$. To this end, consider any $\mathbb{E}_\infty$-ring map $F\colon \mathrm{ko} \to \mathrm{KO}$. The image of the real Bott element is necessarily given by a multiple $n\beta_{\mathbb{R}}$ for some integer $n \in \mathbb{Z}$. For every prime p, completing at p yields an $\mathbb{E}_\infty$-ring map $F^\wedge_p\colon \mathrm{ko}^\wedge_p \to \mathrm{KO}^\wedge_p$. The target is $K(1)$-local and the canonical map $\mathrm{ko}^\wedge_p \to \mathrm{KO}^\wedge_p$ is a $K(1)$-localization (since its fiber is bounded above, hence $K(1)$-acyclic), so $F^\wedge_p$ factors through an $\mathbb{E}_\infty$-ring map $\overline{F^\wedge_p}\colon \mathrm{KO}^\wedge_p \to \mathrm{KO}^\wedge_p$, which is an automorphism since $\mathrm{KO}^\wedge_p$ is Galois over $L_{K(1)}\mathbb{S}$. It acts on β as multiplication by the image of n in $\mathbb{Z}_p$, which is thus a unit in $\mathbb{Z}_p$. This is true for all primes p, hence $n \in \mathbb{Z}^\times$ by the fracture square and F inverts $\beta_{\mathbb{R}}$ as claimed.

In particular, we obtain that also every $\mathbb{E}_\infty$-ring endomorphism of ko is automatically an automorphism. Also, every action of a torsion group G on ko is trivial, because the classifying space BG is rationally acyclic.

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