Higher Presentable Categories as Categories

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These notes are intended to explain how κ-compactly generated (∞, n)-categories in the sense of Stefanich (and a modification due to Aoki) relate to usual (∞, n)-categories, something that is not quite obvious from the slick definition. This material should be regarded as wholly folkloric and known to experts.

Acknowledgements. I would like to thank Maxime Ramzi, Phil Pützstück and Ferdinand Wagner for helpful discussions, as well as Jiacheng Liang, Markus Zetto, Thomas Blom, Shay Ben-Moshe and Rune Haugseng for suggestions.

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1 Higher Presentable Categories as Large Categories

Presentable n-categories[(1)] were originally introduced in [Ste20]. Following a modification due to [Aok25] (see also [Sch26, Lecture III]), we fix an uncountable regular cardinal κ and let $n\mathrm{Pr}^\kappa$ be the category of κ-compactly generated n-categories. These are inductively defined symmetric monoidal κ-compactly generated categories given by $0\mathrm{Pr}^\kappa := \mathrm{An}$ and $(n+1)\mathrm{Pr}^\kappa := \mathrm{Mod}_{n\mathrm{Pr}^\kappa}(\mathrm{Pr}^\kappa)$, where $\mathrm{Pr}^\kappa \hookrightarrow \mathrm{Pr}$ is the (non-full) subcategory on the κ-compactly generated categories and left adjoints[(2)] preserving κ-compact objects. This is well-defined by

[(1)] The prefix '∞-' is consistently dropped henceforth, so 'category' means '$(\infty, 1)$-category' and, more generally, 'n-category' means '(∞, n)-category'.

[(2)] The Pr's appearing are always understood to be variations of Pr^L, but we drop the L from notation.

[Aok25, Section 2.1]; also note the obvious forgetful functor $1\mathrm{Pr}^\kappa \xrightarrow{\sim} \mathrm{Pr}^\kappa$ is an equivalence since An is the tensor unit of Pr^κ. This definition is very slick, though it is not clear at first glance how it relates to the usual n-categories. To this end, Stefanich constructed in [Ste20, Theorem 1.1.2] a comparison functor $n\mathrm{Pr} \to \widehat{n\mathrm{Cat}}$. The purpose of this note is to give an alternative proof of (the κ-compactly generated version of) this theorem and some consequences, relying instead primarily on Heine's work [Hei23], [Hei24] on enriched category theory. The upshot of this approach is that we can identify our comparison functor to be a monomorphism, which does not appear to be immediate from Stefanich's (different) original construction.[3]

To fix the target of this comparison, let $\widehat{n\mathrm{Cat}}$ denote the (huge) category of large n-categories. These are inductively defined via $\widehat{0\mathrm{Cat}} = \widehat{\mathrm{An}}$, the (huge) category of large anima, and $\widehat{(n+1)\mathrm{Cat}} = \widehat{\mathrm{Cat}}_{\widehat{n\mathrm{Cat}}}$, the (huge) category of large $\widehat{n\mathrm{Cat}}$-enriched categories. The theory of enriched categories that we rely on is developed in [GH15]. For comparison of this definition of (large) n-categories with other models (e.g. n-fold Segal, complicial) or the axiomatic approach, see [Hau15], [Lou24] or [BS21] respectively. The categories $\widehat{n\mathrm{Cat}}$ are understood to be symmetric monoidal with their cartesian monoidal structures.

Theorem 1.1. *For any $n \geq 0$, there is a canonical lax symmetric monoidal monomorphism[4] of (huge) symmetric monoidal categories $\iota_n \colon n\mathrm{Pr}^\kappa \hookrightarrow \widehat{n\mathrm{Cat}}$, which*

(a) preserves κ-small limits,

(b) is compatible with the underlying 'enriched graphs' in the sense that the following square commutes:

$$\begin{array}{ccc} n\mathrm{Pr}^\kappa & \xhookrightarrow{\iota_n} & \widehat{n\mathrm{Cat}} \\ \downarrow & & \downarrow \\ \mathrm{Graph}((n-1)\mathrm{Pr}^\kappa) & \xhookrightarrow{(\iota_{n-1})_\circ} & \mathrm{Graph}(\widehat{(n-1)\mathrm{Cat}}). \end{array}$$

Remark 1.2. The condition (b) states that a κ-compactly generated n-category $\mathcal{C}$ and its image $\iota_n\mathcal{C}$ have the same groupoid core and, for any two objects $x, y \in \mathcal{C}$, there is a natural equivalence $\mathrm{Hom}_{\iota_n\mathcal{C}}(x,y) \simeq \iota_{n-1}\mathrm{Hom}_{\mathcal{C}}(x,y)$ of $(n-1)$-categories, where $\mathrm{Hom}_{\mathcal{C}}(x,y) \in (n-1)\mathrm{Pr}^\kappa$ is a priori characterized by the universal property $\mathrm{Map}_{(n-1)\mathrm{Pr}^\kappa}(\mathcal{V}, \mathrm{Hom}_{\mathcal{C}}(x,y)) \simeq \mathrm{Map}_{\mathcal{C}}(\mathcal{V} \otimes x, y)$ for any $\mathcal{V} \in (n-1)\mathrm{Pr}^\kappa$.

Proof. This statement is proven by induction on n, the base case being the inclusion $\mathrm{An} \hookrightarrow \widehat{\mathrm{An}}$. In the inductive step, we consider the composite

$$n\mathrm{Pr}^\kappa \simeq \mathrm{Mod}_{(n-1)\mathrm{Pr}^\kappa}(\mathrm{Pr}^\kappa) \xhookrightarrow{(1)} \mathrm{Mod}_{(n-1)\mathrm{Pr}^\kappa}(\mathrm{Pr}) \overset{(2)}{\simeq} \mathrm{Pr}_{(n-1)\mathrm{Pr}^\kappa} \xhookrightarrow{(2')} \widehat{\mathrm{Cat}}_{(n-1)\mathrm{Pr}^\kappa} \xhookrightarrow{(3)} \widehat{\mathrm{Cat}}_{\widehat{(n-1)\mathrm{Cat}}} \simeq \widehat{n\mathrm{Cat}},$$

where:

- (1) is induced by post-composition with the symmetric monoidal monomorphism $\mathrm{Pr}^\kappa \hookrightarrow \mathrm{Pr}$, which preserves κ-small limits. Then, (1) inherits the structure of a lax symmetric monoidal monomorphism by

[3] This may just betray my own ignorance, however.
[4] This is stronger than just a lax symmetric monoidal functor which is a monomorphism underlying, see Appendix A.

unwinding the convoluted [Lur17, Definition 4.5.1.1] and Proposition A.1.[5] It satisfies (a) since limits in module categories are computed underlying [Lur17, Corollary 4.2.3.3].

- (2) and (2') define a lax symmetric monoidal monomorphism by [Hei23, Theorems 1.2, 1.3][6], where the target inherits its symmetric monoidal structure from $(n-1)\mathrm{Pr}^\kappa$ by [GH15, Corollary 5.7.12]. The property (a) is surely known to experts, but we recall it as Proposition B.3.

- (3) is the change of enrichment along the lax symmetric monoidal functor ι_{n-1} in the sense of [GH15, Corollary 5.7.6]. This refines to a lax symmetric monoidal functor by [Ste20, Remark 2.5.15], which is again a lax symmetric monoidal monomorphism by unwinding [Ste20, Construction 2.5.12]. Furthermore, it satisfies (a) by the inductive hypothesis and since limits in $\widehat{\mathrm{Cat}}_\mathcal{V}$ are created by the forgetful functor to $\mathrm{Graph}(\mathcal{V})$ (again, this is known to experts, but we recall it as Proposition B.2).

  The induced monoidal structure on $\widehat{\mathrm{Cat}}_\mathcal{V}$ is cartesian if the one on $\mathcal{V}$ is cartesian (see also [Ste20, Proposition 2.5.8]). Indeed, if $\mathcal{V},\mathcal{W}$ are monoidal categories admitting terminal objects, there are monoidal localizations $\pi_0\colon \mathcal{V}\times\mathcal{W} \rightleftarrows \mathcal{V}\colon i_0$ resp. $\pi_1\colon \mathcal{V}\times\mathcal{W} \rightleftarrows \mathcal{W}\colon i_1$, which induce adjunctions

$$\pi_{0,*}\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}\times\mathcal{W}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})\colon i_{0,*} \quad \text{resp.} \quad \pi_{1,*}\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}\times\mathcal{W}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})\colon i_{1,*}$$

  by [GH15, Lemma A.5.12] and [Lur17, Proposition 7.3.2.6]. Together with the observation that the external product $\boxtimes$ is the cartesian product on $\mathrm{Alg}_{\mathrm{cat}}$ (see the proof of [GH15, Lemma 5.7.10], or [Hei23, Lemma 7.10]), this yields a natural equivalence $\mathcal{C}\boxtimes\mathcal{D}\simeq i_{0,*}\mathcal{C}\times i_{1,*}\mathcal{D}$ for any $\mathcal{C}\in\widehat{\mathrm{Cat}}_\mathcal{V}, \mathcal{D}\in\widehat{\mathrm{Cat}}_\mathcal{W}$. Taking $\mathcal{V}=\mathcal{W}$ and changing enrichment along the symmetric monoidal, limit-preserving functor $\times\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$, we obtain (invoking Proposition B.2) a canonical equivalence $\mathcal{C}\otimes_\mathcal{V}\mathcal{D}\simeq \times_*(i_{0,*}\mathcal{C}\times i_{1,*}\mathcal{D})\simeq \mathcal{C}\times\mathcal{D}$. Thus, $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ and hence $\mathrm{Cat}_\mathcal{V}$ are cartesian.

The property (b) holds by the very construction of the equivalence in [Hei23], see e.g. Sections 4.2-4.3 in op.cit., and since the functor $\mathrm{Cat}_{(-)}$ of [GH15, Corollary 5.7.6] is compatible with the forgetful functors to graphs (as one functoriality is post- and the other is pre-composition with maps of non-symmetric operads). □

Remark 1.3. In contrast to [Ste20, Theorem 1.1.2(i)], the functors ι_n are *not* compatible with taking underlying categories. This is an artifact of the chosen cardinal cut-off κ. To wit, let $\mathcal{C}\in 2\mathrm{Pr}^\kappa$. The category underlying the 2-category $\iota_2\mathcal{C}$ has mapping anima given by $(\iota_1\,\mathrm{Hom}_\mathcal{C}(x,y))^\simeq$. This is, generally, a large anima, so it cannot agree with the mapping anima in the presentable category $\mathcal{C}$, which is locally small. The precise relationship is recovered by noting $\mathrm{Map}_\mathcal{C}(x,y)\simeq \mathrm{Map}_{\mathrm{Pr}^\kappa}(\mathrm{An},\mathrm{Hom}_\mathcal{C}(x,y))\simeq \mathrm{Hom}_\mathcal{C}(x,y)^\simeq_\kappa$, the anima of κ-compact objects in $\mathrm{Hom}_\mathcal{C}(x,y)$. Thus, $\mathcal{C}$ is a (wide, non-full) subcategory of the category underlying $\iota_2\mathcal{C}$. Furthermore, whilst $\mathcal{C}$ can be recovered from $\iota_2\mathcal{C}$, it can *not* be recovered from the category underlying $\iota_2\mathcal{C}$.

In this context, the statement that the underlying groupoids are equivalent is perhaps slightly surprising: it encodes that any equivalence in $\mathcal{C}$ is automatically a κ-compact object in the corresponding Hom-category.

[5] The functor $\mathrm{Pr}^\kappa\hookrightarrow\mathrm{Pr}$ preserves geometric realizations, in fact all small colimits, so (1) is actually a symmetric monoidal functor. This can be justified by a heavy unwinding of [Lur17, Theorem 4.5.2.1] or [Lur17, Theorem 4.8.5.16].

[6] That it is a lax symmetric monoidal monomorphism is in the more precise statement as [Hei23, Theorem 7.21(2)]. Indeed, his 'lax monoidal inclusion' is our 'lax monoidal monomorphism', by comparing [Hei23, Notation 2.31] and Proposition A.1.

Corollary 1.4. *For any $n \geq 0$, the functor $\iota_n \colon n\mathrm{Pr}^\kappa \to \widehat{n\mathrm{Cat}}$ canonically refines to a functor of (huge) $(n+1)$-categories.*

Proof. This is a general fact about lax (symmetric) monoidal functors of closed (symmetric) monoidal categories, see e.g. [HM24, Lemma C.3.7]. □

Question 1.5. *Is $\iota_n \colon n\mathrm{Pr}^\kappa \to \widehat{n\mathrm{Cat}}$ also a monomorphism of (huge) $(n+1)$-categories?*

Remark 1.6. The forgetful functor $\mathrm{Cat}_\mathcal{V} \to \mathrm{Graph}(\mathcal{V})$ is conservative, so together with Proposition B.2, this implies that it creates monomorphisms. Thus, a monomorphism of $(n+1)$-categories is tested on underlying anima (this is implied by Theorem 1.1) and on the Hom-n-categories, i.e. we would need to check that $\iota_n \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\iota_n\mathcal{C}, \iota_n\mathcal{D})$ is a monomorphism for all κ-compactly generated n-categories $\mathcal{C}, \mathcal{D}$ (here, Fun^L resp. Fun denotes the internal Hom in $n\mathrm{Pr}^\kappa$ resp. $\widehat{n\mathrm{Cat}}$). This would be a significant strenghtening of what Corollary A.5 implies in light of Theorem 1.1.

2 Higher Presentable Categories as Small Categories

Presentable categories are almost always large (any small presentable category is a complete lattice), but they are governed by a small collection of data. This is made precise by Gabriel–Ulmer duality [Lur09b, Propositions 5.5.7.8 & 5.5.7.10], which states that freely adjoining κ-filtered colimits and taking κ-compact objects yields inverse equivalences $\mathrm{Ind}_\kappa(-) \colon \mathrm{Rex}^\kappa \xleftrightarrow{\sim} \mathrm{Pr}^\kappa \colon (-)_\kappa$. Here, $\mathrm{Rex}^\kappa \hookrightarrow \mathrm{Cat}$ is the (non-full) subcategory on (small) categories admitting κ-small colimits and functors preserving those. Furthermore, this equivalence is symmetric monoidal with respect to the Lurie tensor product (for $\mathcal{K} = \{\kappa\text{-small simplicial sets}\}$ on the LHS and $\mathcal{K} = \{\text{small simplicial sets}\}$ on the RHS) by [Lur17, Remark 4.8.1.8] and [Lur09b, Example 5.3.6.8].

Inductively, we construct $n\mathrm{Rex}^\kappa \in \mathrm{CAlg}(\mathrm{Pr}^\kappa)$ by setting $0\mathrm{Rex}^\kappa = \mathrm{An}$ and $n\mathrm{Rex}^\kappa = \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Rex}^\kappa)$ for $n \geq 1$. To see that this is well-defined, we inductively obtain equivalences

$$n\mathrm{Rex}^\kappa = \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Rex}^\kappa) \simeq \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa}(\mathrm{Pr}^\kappa) \simeq \mathrm{Mod}_{(n-1)\mathrm{Pr}^\kappa}(\mathrm{Pr}^\kappa) = n\mathrm{Pr}^\kappa,$$

which we use to equip $n\mathrm{Rex}^\kappa$ with the structure of a commutative algebra in Pr^κ and hence $n\mathrm{Rex}^\kappa_\kappa$ with the structure of a commutative algebra in Rex^κ. We again denote this duality by $\mathrm{Ind}_\kappa(-) \colon n\mathrm{Rex}^\kappa \xleftrightarrow{\sim} n\mathrm{Pr}^\kappa \colon (-)_\kappa$, which is backwards-compatible insofar as it agrees with the usual Gabriel–Ulmer duality on underlying categories. Also note the obvious forgetful functor $1\mathrm{Rex}^\kappa \xrightarrow{\sim} \mathrm{Rex}^\kappa$ is an equivalence since An_κ is the tensor unit of Rex^κ. This gives a 'finitary' perspective on κ-compactly generated n-categories, as in [Sch26, Remark 3.16].

In this section, we now prove a 'finitary' version of Theorem 1.1, which has the upshot of participating in a bona fide adjunction as well as being compatible with passing to cores (in contrast to Remark 1.3). To state this, note that there is an adjunction $\mathbf{PSh}(-)_\kappa \colon \mathrm{An} \rightleftarrows \mathrm{Rex}^\kappa \colon (-)^\simeq$, where the left adjoint is a map in $\mathrm{CAlg}(\mathrm{Pr}^\kappa)$. Inductively, we define the left adjoint $n\mathrm{Rex}^\kappa \to (n+1)\mathrm{Rex}^\kappa$ in $\mathrm{CAlg}(\mathrm{Pr}^\kappa)$ as the base-change along $(n-1)\mathrm{Rex}^\kappa_\kappa \to n\mathrm{Rex}^\kappa_\kappa$ obtained by applying $(-)_\kappa$ to the same map one level lower. The corresponding right adjoints are the core functors and we equip them with their canonical lax symmetric monoidal refinement.

Theorem 2.1. *For any $n \geq 0$, there is a canonical lax symmetric monoidal monomorphism of symmetric monoidal categories $\iota_n \colon n\mathrm{Rex}^\kappa \hookrightarrow n\mathrm{Cat}$, which*

(a) is κ-accessible and admits a (strong) symmetric monoidal left adjoint,

(b) is compatible with the underlying 'enriched graphs' in the sense that the following square commutes:

$$\begin{array}{ccc} n\mathrm{Rex}^\kappa & \xhookrightarrow{\iota_n} & n\mathrm{Cat} \\ \downarrow & & \downarrow \\ \mathrm{Graph}((n-1)\mathrm{Rex}^\kappa) & \xhookrightarrow{(\iota_{n-1})_\circ} & \mathrm{Graph}((n-1)\mathrm{Cat}), \end{array}$$

(c) is compatible with passing to cores in the sense that the following square commutes (in $\mathrm{Cat}^{\otimes,\mathrm{lax}}$):

$$\begin{array}{ccc} n\mathrm{Rex}^\kappa & \xhookrightarrow{\iota_n} & n\mathrm{Cat} \\ \downarrow & & \downarrow{\scriptstyle (-)_{\leq n-1}} \\ (n-1)\mathrm{Rex}^\kappa & \xhookrightarrow{\iota_{n-1}} & (n-1)\mathrm{Cat}. \end{array}$$

Remark 2.2. The left adjoint to $n\mathrm{Pr}^\kappa \simeq n\mathrm{Rex}^\kappa \hookrightarrow n\mathrm{Cat}$ should be considered as taking 'enriched presheaves'.

Proof. This statement is proven by induction on n, the base case being the identity of An. In the inductive step, we consider the composite

$$\begin{array}{ccccc} n\mathrm{Rex}^\kappa \simeq \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Rex}^\kappa) & \xhookrightarrow{(1)} & \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Cat}) & \xhookrightarrow{(2)} & \mathrm{Cat}_{\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)} \\ & \searrow^{(2')} & & & \cup| \\ & & & & \mathrm{Cat}_{(n-1)\mathrm{Rex}^\kappa} \xhookrightarrow{(3)} \mathrm{Cat}_{(n-1)\mathrm{Cat}} \simeq n\mathrm{Cat}, \end{array}$$

where:

- (1) is induced by post-composition with the lax symmetric monoidal monomorphism $\mathrm{Rex}^\kappa \hookrightarrow \mathrm{Cat}$ [Lur17, Corollary 4.8.1.4]. This is again a lax symmetric monoidal monomorphism by the same argument as in Theorem 1.1. Furthermore, there is a (strong) symmetric monoidal left adjoint $F \colon \mathrm{Cat} \to \mathrm{Rex}^\kappa$ [Lur17, Remark 4.8.1.8], so that (1) admits a left adjoint given as the composite

$$\mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Cat}) \xrightarrow{F_\circ} \mathrm{Mod}_{F((n-1)\mathrm{Rex}^\kappa_\kappa)}(\mathrm{Rex}^\kappa) \longrightarrow \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Rex}^\kappa),$$

  where the second functor is base-change along the counit map $F((n-1)\mathrm{Rex}^\kappa_\kappa) \to (n-1)\mathrm{Rex}^\kappa_\kappa$, and this left adjoint admits a (strong) symmetric monoidal refinement, see e.g. [Püt25, Corollary 6.4]. Furthermore, (1) is κ-accessible since $\mathrm{Rex}^\kappa \hookrightarrow \mathrm{Cat}$ is κ-accessible by [Lur09b, Proposition 5.5.7.11] and colimits in module categories are computed underlying [Lur17, Corollary 4.2.3.5].

- (2) is the lax symmetric monoidal monomorphism of [Hei23, Theorem 7.21(1)], and it admits a (strong) symmetric monoidal left adjoint by [Hei24, Corollary 4.47(1)] (taking $\mathcal{V} = (n-1)\mathrm{Rex}^\kappa_\kappa$, $\kappa = \emptyset$, $\mathcal{H}' = \mathrm{L}\emptyset$

and $\mathcal{H} = \emptyset$, where the relevant categories are identified by [Hei24, Example 4.46(1)] and [Hei24, Example 2.123] + [Hei23, Proposition 6.10(2)]). Here, $\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$ is equipped with the Day convolution symmetric monoidal structure.

Furthermore, (2) is κ-accessible: indeed, let $\mathcal{C} \simeq \mathrm{colim}_{i\in\mathcal{I}} \mathcal{C}_i$ be a κ-filtered colimit of $(n-1)\mathrm{Rex}^\kappa_\kappa$-linear categories. This is preserved on underlying anima, since $(-)^\simeq \colon \mathrm{Cat} \to \mathrm{An}$ is accessible, and on Hom-objects we calculate, for any $i \in I$ and $x, y \in X_i$, naturally in $\mathcal{V} \in (n-1)\mathrm{Rex}^\kappa_\kappa$, that (here, $\psi_i \colon \mathcal{C}_i \to \mathcal{C}$ is the structure map)

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(\mathcal{V} \otimes \psi_i(x), \psi_i(y)) &\simeq \mathrm{Map}_{\mathcal{C}}(\psi_i(\mathcal{V} \otimes x), \psi_i(y)) \\ &\simeq \operatorname*{colim}_{(\varphi\colon i\to j)\in\mathcal{I}_{i/}} \mathrm{Map}_{\mathcal{C}_j}(\varphi(\mathcal{V}\otimes x), \varphi(y)) \simeq \operatorname*{colim}_{(\varphi\colon i\to j)\in\mathcal{I}_{i/}} \mathrm{Map}_{\mathcal{C}_j}(\mathcal{V}\otimes\varphi(x), \varphi(y)).\end{aligned}$$

The LHS represents the Hom-object of $\mathcal{C}$ from $\psi_i(x)$ to $\psi_i(y)$ in $\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$, whereas the RHS represents the Hom-object of $\mathrm{colim}_{i\in\mathcal{I}} \mathcal{C}_i$ as computed in $\mathrm{Graph}((n-1)\mathrm{Rex}^\kappa_\kappa)$ by Proposition C.3, hence we can conclude by Corollary C.6.

- The vertical inclusion $\mathrm{Cat}_{(n-1)\mathrm{Rex}^\kappa} \subseteq \mathrm{Cat}_{\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)}$ is the change of enrichment along the restricted Yoneda embedding $(n-1)\mathrm{Rex}^\kappa \subseteq \mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$, which is fully faithful and (strong) symmetric monoidal (e.g. by [Lur17, Corollary 4.8.1.12]). This induced functor is fully faithful underlying by direct inspection, and refines to a (strong) symmetric monoidal functor by [GH15, Corollary 5.7.11] (or [Ste20, Construction 2.5.12]). The Yoneda embedding $(n-1)\mathrm{Rex}^\kappa \subseteq \mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$ is κ-accessible (this is a mere restatement of what κ-compactness is), hence so is the induced functor $\mathrm{Cat}_{(n-1)\mathrm{Rex}^\kappa} \subseteq \mathrm{Cat}_{\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)}$ by combining Corollaries C.3 & C.6.

- The factorization (2′) exists in $\mathrm{Cat}^{\otimes,\mathrm{lax}}$ if and only if it exists underlying in Cat, by Remark A.4. This can be checked on objects, so let $\mathcal{C} \in \mathrm{Mod}_{(n-1)\mathrm{Rex}^\kappa_\kappa}(\mathrm{Rex}^\kappa)$ and $\mathrm{Ind}_\kappa(\mathcal{C}) \in \mathrm{Mod}_{(n-1)\mathrm{Pr}^\kappa}(\mathrm{Pr}^\kappa)$ corresponding under Gabriel–Ulmer duality, where we know the latter has a closed (left) action. Furthermore, there is a fully faithful functor $j \colon \mathcal{C} \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$, which interpolates the respective actions via $\mathrm{Ind}_\kappa(-) \colon (n-1)\mathrm{Rex}^\kappa_\kappa \subseteq (n-1)\mathrm{Rex}^\kappa \simeq (n-1)\mathrm{Pr}^\kappa$. Now, to check that the $\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$-enriched category associated to $\mathcal{C}$ is in fact an $(n-1)\mathrm{Rex}^\kappa$-enriched category, it suffices to check this Hom-wise (by inspecting how the vertical inclusion arises). To this end, let $x, y \in \mathcal{C}$ with images $j(x), j(y) \in \mathrm{Ind}_\kappa(\mathcal{C})$ and consider the Hom-object $\mathrm{Hom}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(x), j(y)) \in (n-1)\mathrm{Pr}^\kappa$. Then we have, naturally in $\mathcal{V} \in (n-1)\mathrm{Rex}^\kappa_\kappa$, that

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(\mathcal{V}\otimes x, y) &\simeq \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(\mathrm{Ind}_\kappa(\mathcal{V})\otimes j(x), j(y)) \\ &\simeq \mathrm{Map}_{(n-1)\mathrm{Pr}^\kappa}(\mathrm{Ind}_\kappa(\mathcal{V}), \mathrm{Hom}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(x), j(y))) \\ &\simeq \mathrm{Map}_{(n-1)\mathrm{Rex}^\kappa}(\mathcal{V}, \mathrm{Hom}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(x), j(y))_\kappa).\end{aligned}$$

In other words, the presheaf $\mathcal{V} \mapsto \mathrm{Map}_{\mathcal{C}}(\mathcal{V}\otimes x, y)$, which is the Hom-object of $\mathcal{C}$ from x to y in $\mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$, is contained in the essential image of $(n-1)\mathrm{Rex}^\kappa \subseteq \mathbf{PSh}((n-1)\mathrm{Rex}^\kappa_\kappa)$ as the image of $\mathrm{Hom}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(x), j(y))_\kappa \in (n-1)\mathrm{Rex}^\kappa$. The functor (2′) then admits a (strong) symmetric monoidal left adjoint by restriction, which is again κ-accessible.

- Lastly, (3) is the change of enrichment along the lax symmetric monoidal monomorphism ι_{n-1}, which is again a lax symmetric monoidal monomorphism by the same argument as in Theorem 1.1. It admits a (strong) symmetric monoidal left adjoint induced by the (strong) symmetric monoidal left adjoint of ι_{n-1} by [Ste20, Remark 2.5.14]. This is again κ-accessible by the same argument as for the vertical inclusion.

The property (b) holds for the same reason as in Theorem 1.1. To check that (c) holds, we proceed by induction. The base case $n = 1$ is clear. In the inductive step, the compatibility of (3) with cores is a consequence of the induction hypothesis. The compatibility of (2) with cores is an instance of its functoriality: the monomorphism $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}) \hookrightarrow \mathrm{Cat}_{\mathbf{PSh}(\mathcal{V})}$ is functorial with respect to restriction along (symmetric) monoidal functors $\mathcal{W} \to \mathcal{V}$ on the LHS and change-of-enrichment along the pullback functor $\mathbf{PSh}(\mathcal{V}) \to \mathbf{PSh}(\mathcal{W})$ on the RHS (essentially by unwinding [Hei23, Lemma 4.65]). Together with noting that $\mathbf{PSh}(-)$ applied to $n\mathrm{Rex}^{\kappa}_{\kappa} \to (n+1)\mathrm{Rex}^{\kappa}_{\kappa}$ restricts to the right adjoint $(n+1)\mathrm{Rex}^{\kappa} \to n\mathrm{Rex}^{\kappa}$ by the Yoneda lemma, this completes the argument. □

3 Change-of-Cardinal

Let $\kappa < \lambda$ be two uncountable regular cardinals. These give rise to commutative triangles

$$\begin{array}{ccc} \mathrm{Pr}^{\kappa} & \hookrightarrow & \mathrm{Pr}^{\lambda} \\ & \searrow & \downarrow \\ & & \widehat{\mathrm{Cat}}, \end{array} \qquad \begin{array}{ccc} \mathrm{Rex}^{\lambda} & \hookrightarrow & \mathrm{Rex}^{\kappa} \\ & \searrow & \downarrow \\ & & \mathrm{Cat}, \end{array}$$

which under Gabriel–Ulmer duality induce an adjunction $\mathrm{Pr}^{\kappa} \rightleftarrows \mathrm{Pr}^{\lambda}$ (or $\mathrm{Rex}^{\kappa} \rightleftarrows \mathrm{Rex}^{\lambda}$), in which both adjoints are monomorphisms.[7] Explicitly, the right adjoint $\mathrm{Pr}^{\lambda} \to \mathrm{Pr}^{\kappa}$ is given by $\mathcal{C} \mapsto \mathrm{Ind}_{\kappa}(\mathcal{C}_{\lambda})$ and the left adjoint $\mathrm{Rex}^{\kappa} \to \mathrm{Rex}^{\lambda}$ is given by freely adjoining λ-small colimits while preserving κ-small colimits.

Note that $\mathrm{Pr}^{\kappa} \hookrightarrow \mathrm{Pr}^{\lambda}$ is a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\lambda})$ (indeed, it preserves λ-compacts since its right adjoint preserves λ-filtered colimits by Gabriel–Ulmer duality and [Lur09b, Proposition 5.5.7.11]). Inductively, we obtain a left adjoint change-of-cardinal functor $n\mathrm{Pr}^{\kappa} \to n\mathrm{Pr}^{\lambda}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\lambda})$ as the composite

$$\mathrm{Mod}_{(n-1)\mathrm{Pr}^{\kappa}}(\mathrm{Pr}^{\kappa}) \hookrightarrow \mathrm{Mod}_{(n-1)\mathrm{Pr}^{\kappa}}(\mathrm{Pr}^{\lambda}) \to \mathrm{Mod}_{(n-1)\mathrm{Pr}^{\lambda}}(\mathrm{Pr}^{\lambda}),$$

where the latter functor is base-change along the left adjoint change-of-cardinal functor one level lower. The corresponding right adjoint change-of-cardinal functor is dually described as the composite

$$\mathrm{Mod}_{(n-1)\mathrm{Rex}^{\lambda}_{\lambda}}(\mathrm{Rex}^{\lambda}) \hookrightarrow \mathrm{Mod}_{(n-1)\mathrm{Rex}^{\lambda}_{\lambda}}(\mathrm{Rex}^{\kappa}) \to \mathrm{Mod}_{(n-1)\mathrm{Rex}^{\kappa}_{\kappa}}(\mathrm{Rex}^{\kappa}),$$

where the latter functor is restriction along $(n-1)\mathrm{Rex}^{\kappa}_{\kappa} \subseteq (n-1)\mathrm{Rex}^{\kappa}_{\lambda} \to (n-1)\mathrm{Rex}^{\lambda}_{\lambda}$, induced by the right adjoint change-of-cardinal functor one level lower. The following aspect is as in the $n = 1$ case.

[7] This should be contrasted with the fact that an adjunction in which both adjoints are fully faithful is automatically an adjoint equivalence.

Proposition 3.1. *Let $\kappa < \lambda$ be two uncountable regular cardinals. The right adjoint change-of-cardinal functor fits into a commutative triangle*

$$\begin{array}{ccc} n\mathrm{Rex}^{\lambda} & \longrightarrow & n\mathrm{Rex}^{\kappa} \\ & \searrow & \downarrow \\ & & n\mathrm{Cat}. \end{array}$$

Proof. This statement is proven by induction on n, the base case having been explained above. In the inductive step, we recall the construction of Theorem 2.1. The compatibility of the right adjoint to the change-of-cardinal functor with (3) is a consequence of the induction hypothesis. The remaining compatibility then follows by the naturality of (2) (as in the proof of Theorem 2.1(c)), since $\mathbf{PSh}(-)$ applied to $(n-1)\mathrm{Rex}^{\kappa}_{\kappa} \to (n-1)\mathrm{Rex}^{\lambda}_{\lambda}$ restricts to the right adjoint $n\mathrm{Rex}^{\lambda} \to n\mathrm{Rex}^{\kappa}$ by the Yoneda lemma. □

Corollary 3.2. *Let $\kappa < \lambda$ be two uncountable regular cardinals. The right adjoint change-of-cardinal functor $n\mathrm{Rex}^{\lambda} \hookrightarrow n\mathrm{Rex}^{\kappa}$ is a monomorphism.* □

The left adjoint change-of-cardinal functor $n\mathrm{Pr}^{\kappa} \to n\mathrm{Pr}^{\lambda}$, however, is *not* compatible with the forgetful functors to $\widehat{n\mathrm{Cat}}$ of Theorem 1.1, already for $n = 2$. Indeed, the composite $2\mathrm{Pr}^{\kappa} \to 2\mathrm{Pr}^{\lambda} \to \widehat{2\mathrm{Cat}} \to \widehat{\mathrm{An}}$ (the latter functor being $(-)^{\simeq}$) maps $1\mathrm{Pr}^{\kappa} \mapsto 1\mathrm{Pr}^{\lambda} \mapsto (\mathrm{Pr}^{\lambda})^{\simeq}$ whereas $2\mathrm{Pr}^{\kappa} \to \widehat{2\mathrm{Cat}} \to \widehat{\mathrm{An}}$ maps $1\mathrm{Pr}^{\kappa} \mapsto (\mathrm{Pr}^{\kappa})^{\simeq}$ and these two (large) anima already differ on π_0. Thus, this is the first instance of the classical picture breaking for $n \geq 2$. In particular, this begs the following question, which we first heard from Maxime Ramzi.

Question 3.3. *Is the left adjoint change-of-cardinal functor $n\mathrm{Pr}^{\kappa} \to n\mathrm{Pr}^{\lambda}$ a monomorphism?*

4 Stefanich Rings as Categorical Spectra

In this section, recall that the category StRing^{κ} of *κ-compactly generated Stefanich rings* is definitionally given as the limit (in $\mathrm{Pr}^{R,\kappa}$, though these are created by the inclusion $\mathrm{Pr}^{R,\kappa} \hookrightarrow \widehat{\mathrm{Cat}}$)

$$\mathrm{StRing}^{\kappa} \quad \simeq \quad \lim_{n\in\mathbb{N}} \left(\dots \longrightarrow \mathrm{CAlg}(n\mathrm{Pr}^{\kappa}) \xrightarrow{\mathrm{End}_{-}(\mathbb{1})} \mathrm{CAlg}((n-1)\mathrm{Pr}^{\kappa}) \longrightarrow \dots \xrightarrow{\mathrm{End}_{-}(\mathbb{1})} \mathrm{CAlg}(0\mathrm{Pr}^{\kappa}) \right).$$

Lemma 4.1. *The endomorphism object functor $\mathrm{End}_{-}(-) \colon n\mathrm{Pr}^{\kappa}_{(n-1)\mathrm{Pr}^{\kappa}/} \to (n-1)\mathrm{Pr}^{\kappa}_{(n-2)\mathrm{Pr}^{\kappa}/}$ is a morphism in $\mathrm{Pr}^{R,\kappa}$ and admits a lax symmetric monoidal refinement that furthermore lifts to $\mathbb{E}_1$-algebras in the target.*

Remark 4.2. This is a more elaborate version of [Aok25, Lemma 2.11].

Proof. For any $n \geq 1$, we have $n\mathrm{Pr}^{\kappa}_{(n-1)\mathrm{Pr}^{\kappa}/} \simeq \mathrm{Alg}_{\mathbb{E}_0}(n\mathrm{Pr}^{\kappa})$ and a map $(n-1)\mathrm{Pr}^{\kappa} \to \mathcal{C}$ of κ-compactly generated n-categories is essentially uniquely determined by specifying a κ-compact object in the κ-compactly generated category underlying $\mathcal{C}$. Then consider the two adjunctions

$$\mathrm{Alg}_{\mathbb{E}_0}(n\mathrm{Pr}^{\kappa}) \underset{}{\overset{\mathrm{Free}_{\mathbb{E}_1/\mathbb{E}_0(-)}}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_1}(n\mathrm{Pr}^{\kappa}) \underset{\mathrm{End}_{-}(-)}{\overset{\mathrm{RMod}_{-}(n\mathrm{Pr}^{\kappa})}{\rightleftarrows}} \mathrm{Mod}_{n\mathrm{Pr}^{\kappa}}(\mathrm{Pr}^{\kappa})_{n\mathrm{Pr}^{\kappa}/} \quad \simeq \quad \mathrm{Alg}_{\mathbb{E}_0}((n+1)\mathrm{Pr}^{\kappa}). \tag{1}$$

The first is a free-forgetful adjunction, see e.g. [Lur17, Corollary 3.1.3.5], and its right adjoint preserves κ-filtered colimits by [Lur17, Proposition 3.2.3.1]. The latter adjunction is obtained by restricting the adjunction $\mathrm{Alg}_{\mathbb{E}_1}(n\mathrm{Pr}^\kappa) \rightleftarrows \mathrm{Mod}_{n\mathrm{Pr}^\kappa}(\mathrm{Pr})_{n\mathrm{Pr}^\kappa/}$ discussed in the paragraph before [Lur17, Corollary 4.8.5.13] to the (non-full) subcategory $\mathrm{Mod}_{n\mathrm{Pr}^\kappa}(\mathrm{Pr}^\kappa)_{n\mathrm{Pr}^\kappa/} \hookrightarrow \mathrm{Mod}_{n\mathrm{Pr}^\kappa}(\mathrm{Pr})_{n\mathrm{Pr}^\kappa/}$ (induced by the symmetric monoidal monomorphism $\mathrm{Pr}^\kappa \hookrightarrow \mathrm{Pr}$, as in the proof of Theorem 1.1), which we now argue is possible.

To this end, it suffices to show that the left adjoint factors through this (non-full) subcategory and that this (non-full) subcategory contains the counit transformation at any of its objects. For this, we write $\mathcal{C} = n\mathrm{Pr}^\kappa$ and only use the property that $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^\kappa)$. To warm up, we recall that the category $\mathrm{RMod}_A(\mathcal{C})$ is κ-compactly generated for any $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ by (the dual of) [Aok25, Lemma 2.1]. More precisely, there is a left adjoint $\mathrm{Free}\colon \mathcal{C} \to \mathrm{RMod}_A(\mathcal{C})$ to the forgetful functor, which preserves κ-compact objects and generates the target under colimits. Furthermore, this left adjoint is canonically (left) $\mathcal{C}$-linear, e.g. because it is the image of the unit map $1_{\mathcal{C}} \to A$ under the functor $\mathrm{RMod}_{-}(\mathcal{C})$, which is valued in (left) $\mathcal{C}$-modules in Pr.

To see that $\mathrm{RMod}_{-}(\mathcal{C})$ factors through $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^\kappa)_{\mathcal{C}/} \hookrightarrow \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr})_{\mathcal{C}/}$, we first check this on objects. For any $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$, we already observed that $\mathrm{RMod}_A(\mathcal{C}) \in \mathrm{Pr}^\kappa$. To check that it is a left module over $\mathcal{C}$ in Pr^κ, i.e. that the action $\mathcal{C} \otimes \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{C})$ preserves κ-compact objects, it suffices to check this on κ-compact generators. For $C, C' \in \mathcal{C}_\kappa$, we have $\mathcal{C} \otimes \mathrm{Free}(\mathcal{C}') \simeq \mathrm{Free}(\mathcal{C} \otimes \mathcal{C}')$ and this is κ-compact, because $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^\kappa)$. Furthermore, the (left) $\mathcal{C}$-linear structure map $\mathcal{C} \to \mathrm{RMod}_A(\mathcal{C})$ selecting the right A-module A is a morphism in Pr^κ, since $A \simeq \mathrm{Free}(1_{\mathcal{C}})$ is κ-compact, again because $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^\kappa)$. Now, we check the factorization on morphisms. For any morphism $A \to B$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$, the left adjoint $\mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_B(\mathcal{C})$ is a morphism in Pr^κ, i.e. it preserves κ-compacts, since the right adjoint preserves κ-filtered colimits, in fact all small colimits, by [Lur17, Corollary 4.2.3.7].

Lastly, for any object $\mathcal{C} \to \mathcal{M}$ in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^\kappa)_{\mathcal{C}/}$, whose structure map selects a κ-compact object $X \in \mathcal{M}$, the counit transformation is given as the essentially unique (left) $\mathcal{C}$-linear map $\mathrm{RMod}_{\mathrm{End}_{\mathcal{M}}(X)}(\mathcal{C}) \to \mathcal{M}$ mapping $\mathrm{End}_{\mathcal{M}}(X) \mapsto X$. To check that this map preserves κ-compact objects, it suffices to check this on the κ-compact generators $\mathrm{Free}(C)$, $C \in \mathcal{C}_\kappa$, but by (left) $\mathcal{C}$-linearity and since the $\mathcal{C}$-action on $\mathcal{M}$ preserves κ-compacts as it is a module in Pr^κ, this reduces to the case $C = 1_{\mathcal{C}}$, which is satisfied by assumption.

Thus, the adjunctions in (1) exist and it remains to argue that both right adjoints admit lax symmetric monoidal refinements. The first right adjoint in fact admits a (strong) symmetric monoidal refinement, as a consequence of [Lur17, Proposition 3.2.4.3]. The second right adjoint admits a lax symmetric monoidal refinement by [Lur17, Corollary 7.3.2.7] as the left adjoint admits a (strong) symmetric monoidal refinement. Indeed, taking $\mathcal{K} = \{\text{small simplicial sets}\}$, the left adjoint $\mathrm{Alg}_{\mathbb{E}_1}(n\mathrm{Pr}^\kappa) \to \mathrm{Mod}_{n\mathrm{Pr}^\kappa}(\mathrm{Cat}(\mathcal{K}))$ admits a symmetric monoidal refinement by [Lur17, Remark 4.8.5.19] that factors through the symmetric monoidal monomorphism $\mathrm{Mod}_{n\mathrm{Pr}^\kappa}(n\mathrm{Pr}^\kappa)^\otimes \hookrightarrow \mathrm{Mod}_{n\mathrm{Pr}^\kappa}(\mathrm{Cat}(\mathcal{K}))^\otimes$ as this can be verified underlying (see Remark A.4), and then we may pass to slices over the unit, i.e. $\mathbb{E}_0$-algebras, again by [Lur17, Proposition 3.2.4.3].

To see that this right adjoint is again κ-accessible, we note filtered colimits in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ are created underlying [Lur17, Proposition 3.2.3.1]. For a κ-filtered colimit $\mathcal{M} \simeq \mathrm{colim}_{i \in \mathcal{I}} \mathcal{M}_i$ in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^\kappa)$ and a κ-compact object

$C \in \mathcal{C}_\kappa$, we check that naturally

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(C, \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{End}_{\mathcal{M}_i}(1)) &\simeq \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Map}_{\mathcal{C}}(C, \mathrm{End}_{\mathcal{M}_i}(1)) \simeq \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Map}_{\mathcal{M}_i}(C\otimes 1, 1)\\ &\simeq \mathrm{Map}_{\mathrm{colim}_{i\in\mathcal{I}}\mathcal{M}_i}(C\otimes 1, 1) \simeq \mathrm{Map}_{\mathcal{C}}(C, \mathrm{End}_{\mathrm{colim}_{i\in\mathcal{I}}\mathcal{M}_i}(1)).\end{aligned}$$

Since $\mathcal{C}$ is κ-compactly generated, the κ-compactly generated objects are jointly conservative and this finishes the proof. □

Lemma 4.3. *There is a canonical equivalence*

$$\mathrm{StRing}^\kappa \quad\simeq\quad \lim_{n\in\mathbb{N}}\left(\dots \xrightarrow{\mathrm{End}_-(-)} n\mathrm{Pr}^\kappa_{(n-1)\mathrm{Pr}^\kappa/} \xrightarrow{\mathrm{End}_-(-)} (n-1)\mathrm{Pr}^\kappa_{(n-2)\mathrm{Pr}^\kappa/} \xrightarrow{\mathrm{End}_-(-)} \dots \xrightarrow{\mathrm{End}_-(-)} 0\mathrm{Pr}^\kappa_{*/}\right).$$

(The limit is taken, equivalently, in $\mathrm{Pr}^{R,\kappa}$ *or* Cat*.)*

Proof. To this end, we construct the following diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(3\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_1}(2\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_2}(1\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_3}(0\mathrm{Pr}^\kappa) \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow \\
\dots & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(3\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(2\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_1}(1\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_2}(0\mathrm{Pr}^\kappa) \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\
\dots & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(3\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(2\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(1\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_1}(0\mathrm{Pr}^\kappa) \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\dots & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(3\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(2\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(1\mathrm{Pr}^\kappa) & \xrightarrow{\mathrm{End}_-(\mathbb{1})} & \mathrm{Alg}_{\mathbb{E}_0}(0\mathrm{Pr}^\kappa).
\end{array}$$

The horizontal maps are induced by $\mathrm{End}_-(\mathbb{1})$ by taking $\mathbb{E}_k$-algebras & the vertical maps are forgetful functors. This diagram is constructed by using Dunn additivity [Lur17, Theorem 5.1.2.2] and Lemma 4.1. Taking vertical limits yields the cosequential diagram that defines StRing^κ, by [Lur17, Corollary 5.1.1.5], whereas taking horizontal limits yields a constant cosequential diagram valued at the limit appearing in the statement of the lemma. Thus, the commutation of limits with limits concludes the proof. □

Corollary 4.4. *There is a canonical monomorphism of (huge) categories* $\iota\colon \mathrm{StRing}^\kappa \hookrightarrow 0\widehat{\mathrm{Cat}}\mathrm{Sp} \subseteq \widehat{\mathrm{Cat}}\mathrm{Sp}$*, which preserves* κ*-small limits.*

Proof. Specializing Theorem 1.1(b), we obtain, for any $n \geq 1$, a commutative square

$$\begin{array}{ccc} n\mathrm{Pr}^{\kappa}_{(n-1)\mathrm{Pr}^{\kappa}/} & \xhookrightarrow{\iota_n} & \widehat{n\mathrm{Cat}}_{*/} \\ \downarrow{\scriptstyle \mathrm{End}_-(-)} & & \downarrow{\scriptstyle \mathrm{End}_-(-)} \\ (n-1)\mathrm{Pr}^{\kappa}_{(n-2)\mathrm{Pr}^{\kappa}/} & \xhookrightarrow{\iota_{n-1}} & \widehat{(n-1)\mathrm{Cat}}_{*/}. \end{array} \tag{2}$$

The horizontal maps are monomorphisms by Theorem 1.1 and Corollary A.6, and κ-small limit-preserving since limits in under-slices are computed underlying. Thus, by Lemma 4.3, taking vertical cosequential limits yields the desired κ-small limit-preserving monomorphism $\iota\colon \mathrm{StRing}^{\kappa} \hookrightarrow \widehat{0\mathrm{Cat}}\mathrm{Sp}$. □

Remark 4.5. It would be preferable if we knew that the square (2) could could be lifted to a commutative square in $\mathrm{Cat}^{\otimes,\mathrm{lax}}$ (as we do know this for the individual morphisms). Then we could directly apply $\mathrm{CAlg}(-)\colon \mathrm{Cat}^{\otimes,\mathrm{lax}} \to \mathrm{Cat}^{\otimes,\mathrm{lax}}$ to these diagrams and take the vertical cosequential limit, thus recovering the original limit description. Furthermore, this would allow us to equip ι with a lax symmetric monoidal structure.

Finally, we derive the 'finitary' variant of Corollary 4.4.

Corollary 4.6. *There is a canonical monomorphism of categories $\iota\colon \mathrm{StRing}^{\kappa} \hookrightarrow 0\mathrm{CatSp} \subseteq \mathrm{CatSp}$, which is κ-accessible and admits a left adjoint.*

Proof. Specializing Theorem 2.1(b), we obtain, for any $n \geq 1$, a commutative square

$$\begin{array}{ccc} n\mathrm{Rex}^{\kappa}_{(n-1)\mathrm{Rex}^{\kappa}_{\kappa}/} & \xhookrightarrow{\iota_n} & n\mathrm{Cat}_{*/} \\ \downarrow{\scriptstyle \mathrm{End}_-(-)} & & \downarrow{\scriptstyle \mathrm{End}_-(-)} \\ (n-1)\mathrm{Rex}^{\kappa}_{(n-2)\mathrm{Rex}^{\kappa}_{\kappa}/} & \xhookrightarrow{\iota_{n-1}} & (n-1)\mathrm{Cat}_{*/}. \end{array}$$

The horizontal maps are monomorphisms by Theorem 2.1 and Corollary A.6, and again admit left adjoints given by applying the left adjoints of the ι_n which preserve the unit objects by (strong) monoidality. Thus, by Gabriel–Ulmer duality and Lemma 4.3, taking vertical cosequential limits yields the desired monomorphism $\iota\colon \mathrm{StRing}^{\kappa} \hookrightarrow \widehat{0\mathrm{Cat}}\mathrm{Sp} \subseteq \mathrm{CatSp}$. It is again κ-accessible and admits a left adjoint by [Lur09b, Proposition 5.5.7.6], since the entire diagram lifts to $\mathrm{Pr}^{R,\kappa}$ by Lemma 4.1 and Theorem 2.1. □

A Lax Symmetric Monoidal Monomorphisms

In this appendix, we study lax symmetric monoidal monomorphisms.

Proposition A.1. *Let $\mathcal{V}^{\otimes} \to \Gamma^{\mathrm{op}}$, $\mathcal{W}^{\otimes} \to \Gamma^{\mathrm{op}}$ be symmetric monoidal categories and $F\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a lax symmetric monoidal functor, i.e. a map of operads. Then the following are equivalent:*

(i) $F^{\otimes}$ is a monomorphism in $\mathrm{Cat}^{\otimes,\mathrm{lax}}$.

(ii) $F^{\otimes}$ is a monomorphism in Op.

(iii) The functor $F^{\otimes}\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ of categories is a monomorphism in Cat.

(iv) The following three conditions hold:

(a) The underlying functor $F\colon \mathcal{V} \to \mathcal{W}$ of categories is a monomorphism in Cat.

(b) The lax structure map $\eta\colon 1_{\mathcal{V}} \to F(1_{\mathcal{W}})$ is 'an epimorphism relative to F', i.e. the composite

$$\mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, v) \xrightarrow{F} \mathrm{Map}_{\mathcal{W}}(F(1_{\mathcal{V}}), F(v)) \xrightarrow{\eta^{\circ}} \mathrm{Map}_{\mathcal{W}}(1_{\mathcal{W}}, F(v))$$

is a monomorphism for all $v \in \mathcal{V}$.

(c) For any $v, v' \in \mathcal{V}$, the lax structure map $\mu_{v,v'}\colon F(v) \otimes_{\mathcal{W}} F(v') \to F(v \otimes_{\mathcal{V}} v')$ in $\mathcal{W}$ is 'an epimorphism relative to F', i.e. the composite

$$\mathrm{Map}_{\mathcal{V}}(v \otimes_{\mathcal{V}} v', v'') \xrightarrow{F} \mathrm{Map}_{\mathcal{W}}(F(v \otimes_{\mathcal{V}} v'), F(v'')) \xrightarrow{\mu^{\circ}_{v,v'}} \mathrm{Map}_{\mathcal{W}}(F(v) \otimes_{\mathcal{W}} F(v'), F(v''))$$

is a monomorphism for all $v'' \in \mathcal{V}$.

Proof. It is clear that (ii) implies (i), since $\mathrm{Cat}^{\otimes,\mathrm{lax}} \subseteq \mathrm{Op}$ is a full subactegory. To prove the converse, consider, for any operad $\mathcal{O}^{\otimes}$, the diagram

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathrm{Op}}(\mathcal{O}^{\otimes}, \mathcal{V}^{\otimes}) & \simeq & \mathrm{Map}_{\mathrm{Cat}^{\otimes}}(\mathrm{Env}(\mathcal{O})^{\otimes}, \mathcal{V}^{\otimes}) & \hookrightarrow & \mathrm{Map}_{\mathrm{Cat}^{\otimes,\mathrm{lax}}}(\mathrm{Env}(\mathcal{O})^{\otimes}, \mathcal{V}^{\otimes}) \\
\downarrow{\scriptstyle F_{\circ}} & & \downarrow{\scriptstyle F_{\circ}} & & \downarrow{\scriptstyle F_{\circ}} \\
\mathrm{Map}_{\mathrm{Op}}(\mathcal{O}^{\otimes}, \mathcal{W}^{\otimes}) & \simeq & \mathrm{Map}_{\mathrm{Cat}^{\otimes}}(\mathrm{Env}(\mathcal{O})^{\otimes}, \mathcal{W}^{\otimes}) & \hookrightarrow & \mathrm{Map}_{\mathrm{Cat}^{\otimes,\mathrm{lax}}}(\mathrm{Env}(\mathcal{O})^{\otimes}, \mathcal{W}^{\otimes}).
\end{array}$$

The horizontal equivalences are afforded by the universal property of Env [Lur17, Proposition 2.2.4.9], and the horizontal monomorphisms are induced by the (non-full) subcategory inclusion $\mathrm{Cat}^{\otimes} \hookrightarrow \mathrm{Cat}^{\otimes,\mathrm{lax}}$. Then, if the right vertical map is a monomorphism, it implies the left vertical map is a monomorphism.

To see that (ii) and (iii) are equivalent, note that $\mathrm{Op} \hookrightarrow \mathrm{Cat}_{/\Gamma^{\mathrm{op}}}$ creates limits and $\mathrm{Cat}_{/\Gamma^{\mathrm{op}}} \to \mathrm{Cat}$ creates weakly contractible limits and is conservative. Thus, the composite forgetful functor $\mathrm{Op} \to \mathrm{Cat}$ preserves and reflects monomorphisms.

To see that (iii) and (iv) are equivalent, note first that (iii) implies (iv)(a) since F is a base-change of $F^{\otimes}$ along $\{\langle 1\rangle\} \hookrightarrow \Gamma^{\mathrm{op}}$. Furthermore, $F^{\otimes}$ induces on groupoid cores the functor

$$(F^{\otimes})^{\simeq} \simeq \coprod_{\langle n\rangle \in \Gamma^{\mathrm{op}}} (F^{\simeq})^{\times n}\colon \coprod_{\langle n\rangle \in \Gamma^{\mathrm{op}}} (\mathcal{V}^{\simeq})^{\times n} \to \coprod_{\langle n\rangle \in \Gamma^{\mathrm{op}}} (\mathcal{W}^{\simeq})^{\times n},$$

which is a monomorphism if and only if F is a monomorphism on groupoid cores. Thus, it remains to show that $F^{\otimes}$ is faithful if and only if (iv)(b) and (iv)(c) hold, assuming (iv)(a).

To this end, consider two objects $(v_1,\dots,v_n),(v'_1,\dots,v'_m)\in\mathcal{V}^{\otimes}$ and the diagram

$$\begin{array}{ccc}\mathrm{Map}_{\mathcal{V}^{\otimes}}((v_1,\dots,v_n),(v'_1,\dots,v'_m)) & \xrightarrow{\quad F^{\otimes}\quad} & \mathrm{Map}_{\mathcal{W}^{\otimes}}((Fv_1,\dots,Fv_n),(Fv'_1,\dots,Fv'_m)) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{\Gamma^{\mathrm{op}}}(\langle n\rangle,\langle m\rangle). & \end{array}$$

That the horizontal map is a monomorphism can be checked on vertical fibers. To this end, let $\phi\colon\langle n\rangle\to\langle m\rangle$ be a map of pointed finite sets. The induced map on fibers is computed, by cocartesian lifting, as the composite

$$\begin{aligned}&\mathrm{Map}_{\mathcal{V}^{\otimes}_{\langle m\rangle}}((\bigotimes_{\phi(i)=1}v_i,\dots,\bigotimes_{\phi(i)=m}v_i),(v'_1,\dots,v'_m))\\ \xrightarrow{F}&\mathrm{Map}_{\mathcal{W}^{\otimes}_{\langle m\rangle}}((F(\bigotimes_{\phi(i)=1}v_i),\dots,F(\bigotimes_{\phi(i)=m}v_i)),(v'_1,\dots,v'_m))\\ \xrightarrow{\mu^{\circ}}&\mathrm{Map}_{\mathcal{W}^{\otimes}_{\langle m\rangle}}((\bigotimes_{\phi(i)=1}F(v_i),\dots,\bigotimes_{\phi(i)=m}F(v_i)),(v'_1,\dots,v'_m)).\end{aligned}$$

Here, the second map is pre-composition with the corresponding lax structure map. This case now decomposes as a product of copies of the case $m=1$, so we may assume $m=1$ w.l.o.g. Then, the case $n=0$ is precisely condition (iv)(b), the case $n=1$ is vacuous and an induction reduces the cases $n\geq 2$ to the case $n=2$, which is precisely condition (iv)(c). □

Example A.2. The inclusion $\mathrm{Cat}^{cc}\hookrightarrow\mathrm{Cat}$ and its restriction $\mathrm{Pr}\hookrightarrow\mathrm{Cat}$ are lax symmetric monoidal monomorphisms by [Lur17, Corollary 4.8.1.4 & Proposition 4.8.1.15].

Remark A.3. Two special cases:

- If F is a symmetric monoidal functor, conditions (iv)(b) and (iv)(c) are implied by condition (iv)(a), so a symmetric monoidal functor is a monomorphism (in, equivalently, $\mathrm{Cat}^{\otimes}$, $\mathrm{Cat}^{\otimes,\mathrm{lax}}$ or Op) if and only if it is a monomorphism underlying.
- If F is fully faithful underlying, condition (iv)(a) is satisfied and condition (iv)(b) resp. (iv)(c) are equivalent to $\eta\colon 1_{\mathcal{V}}\to F(1_{\mathcal{W}})$ resp. $\mu_{v,v'}\colon F(v)\otimes_{\mathcal{W}}F(v')\twoheadrightarrow F(v\otimes_{\mathcal{V}}v')$ for all $v,v'\in\mathcal{V}$ being an actual epimorphism in $\mathcal{W}$.

Remark A.4. The proof also demonstrates how to check whether a lax symmetric monoidal functor $G\colon\mathcal{C}^{\otimes}\to\mathcal{W}^{\otimes}$ of symmetric monoidal categories factors (necessarily essentially uniquely) through $F\colon\mathcal{V}^{\otimes}\hookrightarrow\mathcal{W}^{\otimes}$, namely if and only if the following three conditions hold:

(a) it factors at the level of underlying categories through a (necessarily essentially unique) functor $G'\colon\mathcal{C}\to\mathcal{V}$,

(b) the lax structure map $\eta^{G}\colon 1_{\mathcal{W}}\to G(1_{\mathcal{C}})\simeq FG'(1_{\mathcal{C}})$ is hit by the monomorphism of condition (iv)(b) (the map $1_{\mathcal{V}}\to G'(1_{\mathcal{C}})$ of which it is the image then becomes the lax structure map for the putative lax symmetric monoidal functor $G'\colon\mathcal{C}^{\otimes}\to\mathcal{V}^{\otimes}$),

(c) for any $c,c'\in\mathcal{C}$, the lax structure map $\mu^{G}_{c,c'}\colon FG'(c)\otimes_{\mathcal{W}}FG'(c')\simeq G(c)\otimes_{\mathcal{W}}G(c')\to G(c\otimes_{\mathcal{C}}c')\simeq FG'(c\otimes_{\mathcal{C}}c')$ is hit by the monomorphism of condition (iv)(c) (the necessarily unique map $G'(c)\otimes_{\mathcal{V}}G'(c')\to$

$G'(c \otimes_{\mathcal{C}} c')$ of which it is the image then becomes the lax structure map for the putative lax symmetric monoidal functor $G' \colon \mathcal{C}^{\otimes} \to \mathcal{V}^{\otimes}$).

In particular, if F is (strong) symmetric monoidal and fully faithful underlying, conditions (b) and (c) are automatic (since the monomorphisms in question are equivalences), so the factorization can be tested underlying. Similarly, if F and G are both (strong) symmetric monoidal, conditions (b) and (c) are automatic (since the monomorphisms in question are equivalences), so the factorization can be tested underlying as well.

Corollary A.5. *In the situation of Proposition A.1, if the symmetric monoidal categories are additionally closed, the given conditions are further equivalent to:*

(v) The following three conditions hold:

(a) The underlying functor $F \colon \mathcal{V} \to \mathcal{W}$ of categories is a monomorphism in Cat.

(b) The lax structure map $\eta \colon 1_{\mathcal{V}} \to F(1_{\mathcal{W}})$ is 'an epimorphism relative to F', i.e. the composite

$$\mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, v) \xrightarrow{F} \mathrm{Map}_{\mathcal{W}}(F(1_{\mathcal{V}}), F(v)) \xrightarrow{\eta^{\circ}} \mathrm{Map}_{\mathcal{W}}(1_{\mathcal{W}}, F(v))$$

is a monomorphism for all $v \in \mathcal{V}$.

(c) For any $v, v' \in \mathcal{V}$, the induced map $\psi_{v,v'} \colon F([v,v']_{\mathcal{V}}) \to [F(v), F(v')]_{\mathcal{W}}$ adjoint to the composite

$$F([v,v']_{\mathcal{V}}) \otimes_{\mathcal{W}} F(v) \xrightarrow{\mu_{[v,v']_{\mathcal{V}},v''}} F([v,v']_{\mathcal{V}} \otimes_{\mathcal{V}} v) \xrightarrow{F(\mathrm{ev})} F(v')$$

is 'a monomorphism relative to F', i.e. the composite

$$\mathrm{Map}_{\mathcal{V}}(v'', [v,v']_{\mathcal{V}}) \xrightarrow{F} \mathrm{Map}_{\mathcal{W}}(Fv'', F[v,v']_{\mathcal{V}}) \xrightarrow{(\psi_{v,v'})_{\circ}} \mathrm{Map}_{\mathcal{W}}(Fv'', [Fv, Fv']_{\mathcal{W}})$$

is a monomorphism for all $v'' \in \mathcal{V}$. □

Corollary A.6. *Let $F^{\otimes} \colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a lax symmetric monoidal monomorphism of symmetric monoidal categories. Then the induced functor $F' \colon \mathcal{V}_{1_{\mathcal{V}}/} \to \mathcal{W}_{F(1_{\mathcal{V}})/} \to \mathcal{W}_{1_{\mathcal{W}}/}$ is a monomorphism.*

Proof. This only requires conditions (iv)(a) and (iv)(b) of Proposition A.1. First, we show that F' is faithful. To this end, let $\eta_v \colon 1_{\mathcal{V}} \to v$ and $\eta_{v'} \colon 1_{\mathcal{V}} \to v'$ be two objects of $\mathcal{V}_{1_{\mathcal{V}}/}$ and consider the following diagram of spans

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{V}}(v, v') & \xrightarrow{\eta_v^{\circ}} & \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, v') & \xleftarrow{\eta_{v'}} & \{*\} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \eta^{\circ} \circ F} & & \wr \downarrow \\ \mathrm{Map}_{\mathcal{W}}(F(v), F(v')) & \xrightarrow{(F(\eta_v) \circ \eta)^{\circ}} & \mathrm{Map}_{\mathcal{W}}(1_{\mathcal{W}}, F(v')) & \xleftarrow{F(\eta_{v'}) \circ \eta} & \{*\}, \end{array} \tag{3}$$

where the left horizontal map is a monomorphism by faithfulness of F and the middle vertical map is a monomorphism precisely by condition (iv)(b). Thus, the induced map on pullbacks is a monomorphism, but this map identifies with the induced map $F' \colon \mathrm{Map}_{\mathcal{V}_{1_{\mathcal{V}}/}}(v, v') \to \mathrm{Map}_{\mathcal{W}_{1_{\mathcal{W}}/}}(F(v), F(v'))$, so F' is faithful.

To see that F' is fully faithful on underlying anima, observe that $(\mathcal{C}_{c/})^{\simeq} \simeq \mathcal{C}^{\simeq} \times_{\mathcal{C}} \mathcal{C}_{c/}$ in general. This implies that computing the induced map on mapping anima of groupoid cores amounts to replacing the left vertical map in the diagram (3) with its restriction to components of equivalences, i.e. $\mathrm{Map}_{\mathcal{V}^{\simeq}}(v, v') \xrightarrow{\sim} \mathrm{Map}_{\mathcal{W}^{\simeq}}(F(v), F(v'))$, which is an equivalence as F is a monomorphism. The induced map on pullbacks is then again equivalence (a general fact about maps of cospans whose outer components are equivalences and whose middle component is a monomorphism), completing the proof that F' is a monomorphism. □

B Limits of $\mathcal{V}$-enriched Categories

In this appendix, we study limits of $\mathcal{V}$-enriched categories.

Lemma B.1. *Let $\mathcal{V}$ be a monoidal category admitting small limits and $\pi\colon \mathcal{P}^{\otimes} \to \Delta^{\mathrm{op}}$ be a (generalized) non-symmetric operad. Then, the restriction functor $\mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{P}_{\mathrm{triv}}}(\mathcal{V}) \simeq \mathrm{Fun}(\mathcal{P}_{[1]}, \mathcal{V})$ creates small limits.*

Proof. This is a variant of [Lur17, Corollary 3.2.2.5], but let's prove it. First, note that $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes})$ admits small limits, created by the forgetful functor $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) \to \prod_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Fun}(\mathcal{P}_{[n]}, \mathcal{V}_{[n]})$. Indeed, let $\mathcal{I} \to \mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes})$ be a diagram, $P \in \mathcal{P}^{\otimes}$ and consider the diagram

$$\begin{array}{ccccccc}
\mathcal{I} & \longrightarrow & \mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) & \longrightarrow & \mathrm{Fun}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes}) & \xrightarrow{\mathrm{ev}_P} & \mathcal{V}^{\otimes} \\
\downarrow & \nearrow & \downarrow & \lrcorner & \downarrow & & \downarrow \\
\mathcal{I}^{\triangleleft} & \longrightarrow & \{*\} & \xrightarrow{\pi} & \mathrm{Fun}(\mathcal{P}^{\otimes}, \Delta^{\mathrm{op}}) & \xrightarrow{\mathrm{ev}_P} & \Delta^{\mathrm{op}}.
\end{array}$$

A dashed arrow filling the left square is a limit diagram if and only if the composite filling the first two squares is a relative limit diagram, by [Lur18, Tag 02X6] since $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ is a cocartesian fibration. This is the case if and only if the dashed arrow filling the composite square is a relative limit diagram for all $P \in \mathcal{P}^{\otimes}$, by [Lur18, Tag 02XN], which is the case if and only the corresponding diagram $\mathcal{I}^{\triangleleft} \to \mathcal{V}^{\otimes}_{\pi(P)}$ is a limit diagram for all $P \in \mathcal{P}^{\otimes}$, by another application of [Lur18, Tag 02X6]. Furthermore, such a dashed arrow exists, by [Lur18, Tag 02XG] and since the fiber $\mathcal{V}^{\otimes}_{\pi(P)} \simeq \mathcal{V}^{\times\pi(P)}$ admits small limits by assumption. This proves the claim.

Now, note that the full subcategory $\mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \subseteq \mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes})$ is closed under small limits. Indeed, if $A_\bullet\colon \mathcal{I} \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \subseteq \mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}^{\otimes}, \mathcal{V}^{\otimes})$ is a diagram and $f\colon P \to Q$ is an inert map in $\mathcal{P}^{\otimes}$, then the induced map $(\lim_{\mathcal{I}} A_\bullet)(f)\colon \lim_{\mathcal{I}}^{\pi(P)} A_\bullet(P) \to \lim_{\mathcal{I}}^{\pi(Q)} A_\bullet(Q)$ (here, the superscript denotes the fiber in which the limit is computed) identifies with the composite

$$\lim\nolimits_{\mathcal{I}}^{\pi(P)} A_\bullet(P) \longrightarrow \pi(f)_!(\lim\nolimits_{\mathcal{I}}^{\pi(P)} A_\bullet(P)) \longrightarrow \lim\nolimits_{\mathcal{I}}^{\pi(Q)} \pi(f)_! A_\bullet(P) \quad \simeq \quad \lim\nolimits_{\mathcal{I}}^{\pi(Q)} A_\bullet(Q).$$

Here, the first map is a cocartesian lift of $\pi(f)$, the second map is the natural limit comparison map and the final equivalence holds since $A_\bullet$ factors through $\mathrm{Alg}_{\mathcal{P}}(\mathcal{V})$, i.e. maps preserving inerts. However, since $\pi(f)$ is inert, the cocartesian transport $\pi(f)_!\colon \mathcal{V}^{\otimes}_{\pi(P)} \to \mathcal{V}^{\otimes}_{\pi(Q)}$ identifies by the Segal condition with a corresponding projection $\mathcal{V}^{\times\pi(P)} \to \mathcal{V}^{\times\pi(Q)}$ and thus preserves small limits, so the second map is also an equivalence. In total, the composite is a cocartesian lift of $\pi(f)$, i.e. inert, hence $\lim_{\mathcal{I}} A_\bullet \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{V})$.

Together, this implies that $\mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \to \prod_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Fun}(\mathcal{P}_{[n]}, \mathcal{V}_{[n]})$ creates small limits. For a (generalized) non-symmetric operad map $A\colon \mathcal{P}^{\otimes} \to \mathcal{V}^{\otimes}$, the induced map $A_{[n]}\colon \mathcal{P}_{[n]} \to \mathcal{V}_{[n]}$ identifies with the n-fold cartesian product of $A_{[1]}\colon \mathcal{P}_{[1]} \to \mathcal{V}$ by the Segal condition, hence $\mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \to \mathrm{Fun}(\mathcal{P}_{[1]}, \mathcal{V})$ already suffices. □

Proposition B.2. *Let $\mathcal{V}$ be a monoidal category that admits small limits. Then the forgetful functor $\mathrm{Cat}_{\mathcal{V}} \to \mathrm{Graph}(\mathcal{V})$ creates small limits.*

Proof. The full subcategory $\mathrm{Cat}_{\mathcal{V}} \subseteq \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is reflective by [GH15, Theorem 5.6.6], so its inclusion creates small limits and we may instead prove that the forgetful functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Graph}(\mathcal{V})$ creates small limits. This is a functor of cartesian fibrations over An (cf. [GH15, Definitions 3.6.1, 3.6.4, 4.3.1 & 4.3.3]), where in either case the fibers admits small limits and the cartesian transport maps preserves small limits (in the former case, this holds by Lemma B.1; in the latter case, it is clear). The claim then follows by unwinding the description of limits in the total category obtained from [Lur09b, Corollary 4.3.1.11]. □

Proposition B.3. *Let $\mathcal{V}$ be a presentably monoidal category. Then the inclusion $\mathrm{Pr}_{\mathcal{V}} \hookrightarrow \widehat{\mathrm{Cat}}_{\mathcal{V}}$ creates small limits.*

Proof. First of all, recall that a $\mathcal{V}$-enriched functor is a $\mathcal{V}$-enriched left adjoint if and only if it is a left adjoint underlying and it is $\mathcal{V}$-linear (which is a condition, since $\mathcal{V}$-enriched functors between tensored $\mathcal{V}$-enriched categories are a priori lax $\mathcal{V}$-linear), see e.g. [Hei24, Lemma 2.90], [Ben24, Corollary 4.14].

Now, suppose $\mathcal{C}_{\bullet}\colon \mathcal{I} \to \mathrm{Pr}_{\mathcal{V}}$ is a small diagram and $\mathcal{I}^{\triangleleft} \to \widehat{\mathrm{Cat}}_{\mathcal{V}}$ a limit extension with cone $\mathcal{C} \in \widehat{\mathrm{Cat}}_{\mathcal{V}}$. The underlying diagram of categories is a limit diagram (e.g. by Proposition B.2, since it is the change-of-enrichment along the limit-preserving functor $\mathrm{Map}_{\mathcal{V}}(1,-)\colon \mathcal{V} \to \mathrm{An}$) and thus factors through a limit diagram in Pr by [Lur09b, Proposition 5.5.3.13]. For the limit object $\mathcal{C}$ to be tensored as a $\mathcal{V}$-enriched category is to ask that $\mathrm{Hom}_{\mathcal{C}}(x,-)\colon \mathcal{C} \to \mathcal{V}$ admits a left adjoint for all $x = (x_i)_{i\in\mathcal{I}} \in \mathcal{C}$. This left adjoint is induced as the limit over the left adjoints $(-)\otimes x_i\colon \mathcal{V} \to \mathcal{C}_i$ (these assemble into a coherent diagram by $\mathcal{V}$-linearity) and applying adjoint descent [HY17, Theorem 5.5], using Proposition B.2 once more to identify the limit with $\mathrm{Hom}_{\mathcal{C}}(x,-)$. Thus, $\mathcal{C} \in \mathrm{Pr}_{\mathcal{V}}$ and the projections $\mathcal{C} \to \mathcal{C}_i$ are also $\mathcal{V}$-linear by construction, i.e. the diagram factors through $\mathrm{Pr}_{\mathcal{V}}$.

To check that the diagram has the correct universal property in $\mathrm{Pr}_{\mathcal{V}}$, it then suffices to show that a functor $\mathcal{D} \to \mathcal{C}$ from a presentable $\mathcal{V}$-enriched category $\mathcal{C}$ is a $\mathcal{V}$-enriched left adjoint if and only if all the component functors $\mathcal{D} \to \mathcal{C}_i$, $i \in I$ are $\mathcal{V}$-enriched left adjoints. This is clear for the underlying functors being left adjoint, again by [Lur09b, Proposition 5.5.3.13], and it also holds for the $\mathcal{V}$-linearity condition since the projections are $\mathcal{V}$-linear and jointly conservative. □

C Filtered Colimits of $\mathcal{V}$-enriched categories

In this appendix, we study filtered colimits of $\mathcal{V}$-enriched categories. To analyze these, we utilize the Segal presheaf model of Gepner–Haugseng [GH15, Section 4.5].

Proposition C.1. *The following are true:*

(1) Let $\mathcal{V}$ be a cocomplete category resp. a κ-compactly generated category. The category $\mathrm{Graph}(\mathcal{V})$ is canonically equivalent to the full subcategory of $\mathbf{PSh}(\mathrm{colim}^{\mathrm{oplax}}[\mathcal{V} \rightrightarrows *])$ resp. $\mathbf{PSh}(\mathrm{colim}^{\mathrm{oplax}}[\mathcal{V}^{\kappa} \rightrightarrows *])$ on

those presheaves ϕ satisfying the representability condition: the induced functor $\mathcal{V}^{\mathrm{op}} \to \mathrm{An}_{/\phi(*)^{\times 2}}$ resp. $\mathcal{V}^{\kappa,\mathrm{op}} \to \mathrm{An}_{/\phi(*)^{\times 2}}$ preserves small resp. κ-small limits.

(2) *Let $\mathcal{V}$ be a presentably resp. κ-compactly generatedly monoidal category. The category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is canonically equivalent to the full subcategory of the presheaf category $\mathbf{PSh}((\mathcal{V}^{\amalg}_{\otimes}))$ resp. $\mathbf{PSh}((\mathcal{V}^{\amalg}_{\otimes})^{\kappa})$ on those presheaves ϕ satisfying the condition of (1) as well as the following Segal condition: for every object $X \in \mathcal{V}^{\amalg}_{\otimes}$ resp. $X \in (\mathcal{V}^{\amalg}_{\otimes})^{\kappa}$ lying over $[n] \in \Delta$, the canonical square*

$$\begin{array}{ccc} \phi(X) & \longrightarrow & \phi(X|_{\{0,\dots,n-1\}}) \\ \downarrow & \lrcorner & \downarrow \\ \phi(X|_{\{n-1,n\}}) & \longrightarrow & \phi(*) \end{array}$$

is cartesian. □

Remark C.2. There is a canonical map $\mathrm{colim}^{\mathrm{oplax}}[\mathcal{V} \rightrightarrows *] \to \mathcal{V}^{\amalg}_{\otimes}$ resp. $\mathrm{colim}^{\mathrm{oplax}}[\mathcal{V}^{\kappa} \rightrightarrows *] \to (\mathcal{V}^{\amalg}_{\otimes})^{\kappa}$, restricting along which induces the forgetful functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Graph}(\mathcal{V})$ under the above equivalences.

Proposition C.3. *Let $\mathcal{V}$ be a κ-compactly generated category and $(X_\bullet, \Phi_\bullet)\colon \mathcal{I} \to \mathrm{Graph}(\mathcal{V})$ a filtered diagram. The colimit of this diagram is given by the anima $X \simeq \mathrm{colim}_{i\in\mathcal{I}} X_i$ and the functor $\Phi\colon X \times X \to \mathcal{V}$ induced by*

$$X_i \times X_i \to \mathcal{V},\ (x,y) \mapsto \operatorname*{colim}_{(\varphi\colon i\to j)\in\mathcal{I}_{i/}} \Phi_j(\varphi(x),\varphi(y)), \qquad i \in \mathcal{I}.$$

Proof. Identifying $\mathrm{Graph}(\mathcal{V})$ with the full subcategory of $\mathbf{PSh}(\mathrm{colim}^{\mathrm{oplax}}[\mathcal{V}^{\kappa} \rightrightarrows *])$ given by Proposition C.1, we note that it is stable under κ-filtered colimits since these commute with κ-small limits in An, hence κ-filtered colimits can be computed pointwise. Let $\mathcal{F}_\bullet\colon \mathcal{I} \to \mathbf{PSh}(\mathrm{colim}^{\mathrm{oplax}}[\mathcal{V}^{\kappa} \rightrightarrows *])$ denote the corresponding diagram. The underlying anima is given by evaluation at $([0], *)$, i.e. it is $X \simeq \mathrm{colim}_{i\in\mathcal{I}} X_i$.

To compute Φ, we note that the functor $\mathcal{V}^{\kappa,\mathrm{op}} \to \mathrm{An}_{/X\times X}$ is thus given by $v \mapsto (\mathrm{colim}_{i\in\mathcal{I}} \mathcal{F}_i(v) \to X \times X)$ obtained as the colimit of the maps $\mathcal{F}_i(v) \to X_i \times X_i$ over $i \in \mathcal{I}$. Identifying $\mathrm{An}_{/X\times X} \simeq \mathrm{Fun}(X \times X, \mathrm{An})$ by taking fibers, we curry this to obtain the functor $X \times X \to \mathrm{Fun}(\mathcal{V}^{\kappa,\mathrm{op}}, \mathrm{An})$, which factors through the restricted Yoneda embedding as $\Phi\colon X \times X \to \mathrm{An}$. For $(x_i, y_i) \in X_i \times X_i$ with image $(x, y) \in X \times X$, we thus obtain

$$\Phi(x,y) \simeq \left[\operatorname*{colim}_{i\in\mathcal{I}} \mathcal{F}_i(V)\right] \times_{X\times X} \{(x,y)\} \simeq \operatorname*{colim}_{(\varphi\colon i\to j)\in\mathcal{I}_{i/}} \left[\mathcal{F}_j(v) \times_{X_j\times X_j} \{(\varphi(x_i),\varphi(y_i))\}\right].$$

This is a κ-filtered colimit, so this functor in v is represented by $\mathrm{colim}_{(\varphi\colon i\to j)\in\mathcal{I}_{i/}} \Phi_j(\varphi(x_i),\varphi(y_i))$, as desired. □

Proposition C.4. *Let $\mathcal{V}$ be a κ-compactly generatedly monoidal category, i.e. $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\kappa})$. Then the forgetful functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Graph}(\mathcal{V})$ creates κ-filtered colimits.*

Proof. Let's consider the commutative diagram provided by Proposition C.1:

$$\begin{array}{ccccc} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) & \simeq & \mathbf{PSh}((\mathcal{V}^{\wedge}_{\otimes})^{\kappa})^{\mathrm{Seg,rep}} & \subseteq & \mathbf{PSh}((\mathcal{V}^{\wedge}_{\otimes})^{\kappa}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Graph}(\mathcal{V}) & \simeq & \mathbf{PSh}(\mathrm{colim}^{\mathrm{oplax}}\,[\mathcal{V}^{\kappa} \rightrightarrows *])^{\mathrm{rep}} & \subseteq & \mathbf{PSh}(\mathrm{colim}^{\mathrm{oplax}}\,[\mathcal{V}^{\kappa} \rightrightarrows *]). \end{array}$$

To prove that the left vertical functor creates κ-filtered colimits, it suffices to prove that the horizontal inclusions are stable under κ-filtered colimits, since the right vertical functor preserves κ-filtered colimits. The argument is the same in both cases: the representability resp. Segal condition is stable under κ-filtered colimits as it is expressed in terms of κ-small limits resp. pullbacks and these commute with κ-filtered colimits in An. □

Remark C.5. In fact, Proposition C.3 holds for all cocomplete categories $\mathcal{V}$, and Proposition C.4 holds for all presentably monoidal $\mathcal{V}$ and all filtered colimits. These statements, however, are more finicky to prove.

Corollary C.6. *Let $\mathcal{V}$ be a κ-compactly generatedly monoidal category, i.e. $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\kappa})$. Then the forgetful functor $\mathrm{Cat}_{\mathcal{V}} \to \mathrm{Graph}(\mathcal{V})$ creates κ-filtered colimits.*

Proof. The forgetful functor $U\colon \mathcal{V} \to \mathrm{An}$ represented by the unit $1_{\mathcal{V}} \in \mathcal{V}$ preserves κ-filtered colimits as the unit is κ-compact. Thus, the forgetful functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathrm{An}) \simeq \mathrm{Seg}(\mathrm{An})$ preserves κ-filtered colimits by combining Proposition C.4 and Proposition C.3. Then the right vertical map in the cartesian square

$$\begin{array}{ccc} \mathrm{Cat}_{\mathcal{V}} & \longrightarrow & \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Cat} & \longrightarrow & \mathrm{Seg}(\mathrm{An}) \end{array}$$

preserves κ-filtered colimits and the lower horizontal map preserves (κ-)filtered colimits by [Lur09a, Proposition 1.2.29]. Thus, $\mathrm{Cat}_{\mathcal{V}} \subseteq \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ preserves κ-filtered colimits by [Lur09b, Lemma 5.4.5.5], hence so does the composite $\mathrm{Cat}_{\mathcal{V}} \to \mathrm{Graph}(\mathcal{V})$ by Proposition C.4. It then also creates these colimits as it is conservative. □

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